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The zero-sum constant, the Davenport constant and their analogues

Maciej Zakarczemny

mzakarczemny@pk.edu.pl | http://orcid.org/0000-0002-7202-1244

Department of Applied Mathematics, Faculty of Computer Science and Telecommunications, Cracow University of Technology

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Abstract

Let $\mathsf{D}(G)$ be the Davenport constant of a finite Abelian group G. For a positive integer m (the case $m=1$, is the classical case) let $\mathsf{E}_m(G)$ (or $\eta_m(G)$) be the least positive integer t such that every sequence of length t in G contains m disjoint zero-sum sequences, each of length $|G|$ (or of length $\leq \exp(G)$, respectively). In this paper, we prove that if G is an Abelian group, then $\mathsf{E}_m(G)=\mathsf{D}(G)-1+m|G|$, which generalizes Gao's relation. Moreover, we examine the asymptotic behaviour of the sequences $(\mathsf{E}_m(G))_{m\geq 1}$ and $(\eta_m(G))_{m\geq 1}$. We prove a generalization of Kemnitz's conjecture. The paper also contains a result of independent interest, which is a stronger version of a result by Ch. Delorme, O. Ordaz, D. Quiroz. At the end, we apply the Davenport constant to smooth numbers and make a natural conjecture in the non-Abelian case.

Keywords: zero-sum sequence, Davenport constant, finite Abelian group

1. Introduction

We will define and investigate some generalizations of the Davenport constant, see (Alon & Dubiner, 1995; Edel et al., 2007; Freeze & Schmid, 2010; Gao & Geroldinger, 2006; Geroldinger & Halter-Koch, 2006; Olson, 1969a, 1969b; Reiher, 2007; Rogers, 1963). Davenport's constant is connected with algebraic number theory as follows. For an algebraic number field K, let $\mathcal{O}_K$ be its ring of integers and G the ideal class group of $\mathcal{O}_K$. Let $x \in \mathcal{O}_K$ be an irreducible element. If $\mathcal{O}_K$ is a Dedekind domain, then $x\mathcal{O}_K = \prod_{i=1}^{r} P_i$, where P_i are prime ideals in $\mathcal{O}_K$ (not necessarily distinct). The Davenport constant $\mathsf{D}(G)$ is the maximal number of prime ideals P_i (counted with multiplicities) in the prime ideal decomposition of the integral ideal $x\mathcal{O}_K$, see (Halter-Koch, 1992; Olson, 1969a).

The precise value of the Davenport constant is known, among others, for p-groups and for groups of rank at most two. The determination of $\mathsf{D}(G)$ for general finite Abelian groups is an open question, see (Girard, 2018).

2. General notation

Let $\mathbb{N}$ denote the set of positive integers (natural numbers).

We set $[a, b] = \{x: a \le x \le b, x \in \mathbb{Z}\}$, where $a, b \in \mathbb{Z}$. Our notation and terminology is consistent with (Geroldinger & Ruzsa, 2009). Let G be a non-trivial additive finite Abelian group. G can be uniquely decomposed as a direct sum of cyclic groups $C_{n_1} \oplus C_{n_2} \oplus \ldots \oplus C_{n_r}$ with natural numbers $1 < n_1 | n_2 | \ldots | n_r$. The number r of summands in the above decomposition of G is expressed as $r = r(G)$ and called the rank of G. The integer n_r is called the exponent of G and denoted by $\exp(G)$. In addition, we define $\mathsf{D}^*(G)$ as $\mathsf{D}^*(G) = 1 + \sum_{i=1}^{r}(n_i - 1)$. We write any finite sequence S of l elements of G in the form $\prod_{g \in G} g^{v_g(S)} = g_1 \cdot g_2 \cdot \ldots \cdot g_l$ (this is a formal Abelian product), where l is the length of S denoted by $|S|$, and $v_g(S)$ is the multiplicity of g in S. S corresponds to the sequence (in the traditional sense) $(g_1, g_2, \ldots, g_l)$, where we forget the ordering of the terms. By $\sigma(S)$ we denote the sum of $S: \sigma(S) = \sum_{g \in G} v_g(S) g \in G$.

The Davenport constant $\mathsf{D}(G)$ is defined as the smallest $t \in \mathbb{N}$ such that each sequence over G of length at least t has a non-empty zero-sum subsequence. Equivalently, $\mathsf{D}(G)$ is the maximal length of a zero-sum sequence of elements of G and with no proper zero-sum subsequence. One of the best bounds for $\mathsf{D}(G)$ known so far is:

$$\mathsf{D}^*(G) \le \mathsf{D}(G) \le n_r \left(1 + \log \frac{|G|}{n_r}\right). \tag{1}$$

The small Davenport constant $\mathsf{d}(G)$ is the maximal length of a zero-sum free sequence over $|G|$. If $|G|$ is a finite Abelian group then $\mathsf{d}(G) = \mathsf{D}(G) - 1$, see (Geroldinger & Ruzsa, 2009 [Definition 2.1.1.]). Alford, Granville and Pomerance (1994) used the bound (1) to prove the existence of infinitely many Carmichael numbers. Dimitrov (2007) used the Alon Dubiner constant (Alon & Dubiner, 1995) to prove the inequality:

$$\frac{\mathsf{D}(G)}{\mathsf{D}^*(G)} \le (Kr \log r)^r,$$

for an absolute constant K. It is known that for groups of rank at most two and for p-groups, where p is a prime, the left hand side inequality (1) is in fact an equality, see (Olson, 1969a, 1969b). This result suggests that $\mathsf{D}^*(G) = \mathsf{D}(G)$. However, there are infinitely many groups G with rank $r > 3$ such that $\mathsf{D}(G) > \mathsf{D}^*(G)$. There are more recent results on groups where the Davenport constant does not match the usual

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lower bound, see (Geroldinger & Schneider, 1992). The following Remark 2.1 lists some basic facts for the Davenport constant, see (Delorme, Ordaz & Quiroz, 2001; Geroldinger & Schneider, 1992; Schmid, 2011; Sheikh, 2017).

Remark 2.1. Let G be a finite additive Abelian group.

1. Then $\mathsf{D}(G)=\mathsf{D}^*(G)$ in each of the following cases:
   - G is a p-group;
   - G has rank $r \le 2$;
   - $G=C_p \oplus C_p \oplus C_{p^n m}$, with p a prime number, $n \ge 2$ and m a natural number coprime with p^n (more generally, if $G=G_1 \oplus C_{p^k n}$, where G_1 is a p-group and $p^k \ge \mathsf{D}^*(G_1)$);
   - $G=C_2 \oplus C_2 \oplus C_2 \oplus C_{2n}$, with odd n;
   - $G=C_{2p^{k_1}} \oplus C_{2p^{k_2}} \oplus C_{2p^{k_3}}$, with p prime, $k_1, k_2, k_3 \ge 0$;
   - $G=C_2 \oplus C_{2n} \oplus C_{2nm}$, with n, m natural numbers;
   - G has one of the forms $C_3 \oplus C_3 \oplus C_{3n}$, $C_4 \oplus C_4 \oplus C_{4n}$, or $C_6 \oplus C_6 \oplus C_{6n}$, with n a natural number;
   - $G=C_5 \oplus C_5 \oplus C_{10}$.

2. Then $\mathsf{D}(G)>\mathsf{D}^*(G)$ in each of the following cases:
   - $G=C_n \oplus C_n \oplus C_n \oplus C_{2n}$, with odd $n \ge 3$;
   - $G=C_2^{r-1} \oplus C_{2n}$, with $r \ge 5$ and odd $n \ge 3$;
   - $G=C_3 \oplus C_9 \oplus C_9 \oplus C_{18}$;
   - $G=C_3 \oplus C_{15} \oplus C_{15} \oplus C_{30}$;
   - let $n \ge 2$, $k \ge 2$, $(n,k)=1, 0 \le \rho \le n-1$, and $G=C_n^{(k-1)n+\rho} \oplus C_{kn}$.
     If $\rho \ge 1$ and $\rho \not\equiv n \pmod k$, then $\mathsf{D}(G) \ge \mathsf{D}^*(G)+\rho$.
     If $\rho \le n-2$ and $x(n-1-\rho) \not\equiv n \pmod k$ for any $x \in [1, n-1]$, then $\mathsf{D}(G) \ge \mathsf{D}^*(G)+\rho+1$.

3. Definitions

In this section, we will provide some definitions of classical invariants. We begin with some notations and remarks that will be used throughout the paper.

Definition 3.1. Let G be a finite Abelian group, and m, k be positive integers such that $k \ge \exp(G)$, and $\varnothing \ne I \subseteq \mathbb{N}$.

1. By $\mathsf{s}_I(G)$ we denote the smallest $t \in \mathbb{N} \cup \{\infty\}$ such that every sequence S (with repetition allowed) over G of length t contains a non-empty subsequence S' such that $\sigma(S')=0$, $|S'| \in I$. We use notation $\mathsf{s}_{\le k}(G)$ or $\mathsf{D}^k(G)$ to denote $\mathsf{s}_I(G)$ if $I=[1,k]$, see (Balasubramanian & Bhowmik, 2006; Chintamani et al., 2012; Delorme, Ordaz, & Quiroz, 2001).
2. By $\mathsf{s}_{I,m}(G)$ we denote the smallest $t \in \mathbb{N} \cup \{\infty\}$ such that every sequence S over G of length t contains at least m disjoint non-empty subsequences $S_1, S_2, \dots, S_m$ such that $\sigma(S_i)=0, |S_i| \in I$.
3. Let $\mathsf{E}(G) := \mathsf{s}_{\{|G|\}}(G)$, i.e. the smallest $t \in \mathbb{N} \cup \{\infty\}$ such that every sequence S over G with length t contains non-empty subsequence S' such that $\sigma(S')=0$, $|S'|=|G|$. Note that $\mathsf{E}(G)$ is the classical zero-sum constant.
4. Let $\mathsf{E}_m(G) := \mathsf{s}_{\{|G|\},m}(G)$, i.e. the smallest $t \in \mathbb{N} \cup \{\infty\}$ such that every sequence S over G with length t contains at least m non-empty subsequences $S_1, S_2, \dots, S_m$ such that $\sigma(S_i)=0$, $|S_i|=|G|$.
5. Also, we define $\eta(G) := \mathsf{s}_{\le \exp(G)}(G)$, $\mathsf{s}(G) := \mathsf{s}_{\{\exp(G)\}}(G)$, $\mathsf{D}_m(G) := \mathsf{s}_{\mathbb{N},m}(G)$ (see (Halter-Koch, 1992)), $\mathsf{s}_{(m)}(G) := \mathsf{s}_{\{\exp(G)\},m}(G)$.

Remark 3.2. For $n, n' \in \mathbb{N}$, $\varnothing \ne I \subseteq \mathbb{N}$, by definition

$\mathsf{s}_{\mathbb{N}}(G)=\mathsf{s}_{[1,D(G)]}(G)=\mathsf{D}(G)=\mathsf{D}_1(G)$,
$\mathsf{s}_{I,1}(G)=\mathsf{s}_I(G)$, $\mathsf{s}_{[1,D(G)],n}(G)=\mathsf{D}_n(G)$, $\mathsf{s}_{\{|G|\},n}=\mathsf{E}_n(G)$.

Note that $\mathsf{s}(G) := \mathsf{s}_{\{\exp(G)\}}(G)$ is the classical Erdös-Ginzburg-Ziv constant (Fan, Gao, & Zhong, 2011). We call $\mathsf{s}_{(m)}(G) = \mathsf{s}_{\{\exp(G)\},m}(G)$ the m-wise Erdös-Ginzburg--Ziv constant of G. Thus, in this notation $\mathsf{s}_{(1)}(G) = \mathsf{s}(G)$.
One can derive, for example, the following inequalities.

Remark 3.3. If $n' \geq n$, then $\mathsf{s}_{I,n'}(G) \geq \mathsf{s}_{I,n}(G)$. If $\mathsf{D}(G) \geq k \geq k' \geq \exp(G)$, then $\mathsf{s}_{[1,\exp(G)],n}(G) \geq \mathsf{s}_{[1,k'],n}(G) \geq \mathsf{s}_{[1,k],n}(G) \geq \mathsf{D}_n(G)$.

If $\mathsf{D}(G) \geq k \geq \exp(G)$, then $\eta(G) \geq \mathsf{s}_{\leq k}(G) \geq \mathsf{D}(G)$. If $k \geq k'$, then

$$\mathsf{s}_{\leq k'}(G) \geq \mathsf{s}_{\leq k}(G). \tag{2}$$

We note that a sequence S over G of length $|S| \geq m\mathsf{D}(G)$ can be partitioned into m disjoint subsequences S_i of length $|S_i| \geq \mathsf{D}(G)$. Thus, each S_i contains a non-empty zero-sum subsequence. Hence $\mathsf{D}_m(G) \leq m\mathsf{D}(G)$. See also (Halter--Koch, 1992 [Proposition 1 (ii)]).

4. The m-wise zero-sum constant and the m-wise Erdös--Ginzburg-Ziv constant of G

In 1996, Gao and Caro independently proved that

$$\mathsf{E}(G) = \mathsf{D}(G) + |G| - 1 \tag{3}$$

for any finite Abelian group, see (Caro, 1996; Gao, 1995, 1996). For a proof in modern language we refer to (Geroldinger & Halter-Koch, 2006 [Proposition 5.7.9]); see also (Delorme, Ordaz & Quiroz, 2001; Gao, 1994; Hamidoune, 1996). Relation (3) unifies research on constants $\mathsf{D}(G)$ and $\mathsf{E}(G)$. We start this section with the result that can be used to unify research on constants $\mathsf{E}(G)$ and $\mathsf{E}_m(G)$.

Theorem 4.1. If G is a finite Abelian group of order $|G|$, then

$$\mathsf{E}_m(G) = \mathsf{E}(G) + (m-1)|G| = \mathsf{D}(G) - 1 + m|G| = \mathsf{d}(G) + m|G|. \tag{4}$$

Proof. By (3) and (Geroldinger & Ruzsa, 2009 [Lemma 2.1.2.]) we obtain

$$\mathsf{E}(G) + (m-1)|G| = \mathsf{D}(G) + m|G| - 1 = \mathsf{d}(G) + m|G|.$$

Let $S = a_1 \cdot a_2 \cdot \ldots \cdot a_{D(G)-1}$ be a sequence of $\mathsf{D}(G) - 1$ non-zero elements in G.

Using the definition of $\mathsf{D}(G)$, we may assume that S does not contain any non-empty subsequence S' such that $\sigma(S') = 0$. We put

$$T = a_1 \cdot a_2 \cdot \ldots \cdot a_{D(G)-1} \cdot \underbrace{0 \cdot \ldots \cdot 0}_{m|G|-1 \text{ times}}.$$

We observe that the sequence T does not contain m disjoint non-empty subsequences $T_1, T_2, \ldots, T_m$ such that $\sigma(T_i) = 0$ and $|T_i| = |G|$ for $i \in [1, m]$. This implies that $\mathsf{E}_m(G) > \mathsf{D}(G) + m|G| - 2$. Hence

$$\mathsf{E}_m(G) \geq \mathsf{E}(G) + (m-1)|G|.$$

On the other hand, if S is any sequence such that $|S| \geq \mathsf{E}(G) + (m-1)|G|$, then one can sequentially extract at least m disjoint subsequences $S_1, \ldots, S_m$, such that $\sigma(S_i) = 0$ in G and $|S_i| = |G|$. Thus

$$\mathsf{E}_m(G) \leq \mathsf{E}(G) + (m-1)|G|. \qquad \blacksquare$$

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Corollary 4.2. For every finite Abelian group, the sequence $(\mathsf{E}_m(G))_{m\geq 1}$ is an arithmetic progression with difference $|G|$.

Corollary 4.3. If p is a prime and $G = C_{p^{e_1}} \oplus \ldots \oplus C_{p^{e_k}}$ is a p-group, then for a natural m we have $E_m(G) = mp^{\sum_{i=1}^{k} e_i} + \sum_{i=1}^{k}\left(p^{e_i} - 1\right)$.

Proof. It follows from Remark 2.1. and Theorem 4.1. ■

We recall that by $\mathsf{s}_{(m)}(G)$ we denote the smallest $t \in \mathbb{N} \cup \{\infty\}$ such that every sequence S over G of length t contains at least m disjoint non-empty subsequences $S_1, S_2, \ldots, S_m$ such that $\sigma(S_i) = 0$, $|S_i| = \exp(G)$.

Theorem 4.4. If G is a finite Abelian group, then

$$\eta(G) + m\exp(G) - 1 \leq \mathsf{s}_{(m)}(G) \leq \mathsf{s}(G) + (m-1)\exp(G).$$

Proof. The proof runs along the same lines as the proof of Theorem 4.1.

Let $S = a_1 \cdot a_2 \cdot \ldots \cdot a_{\eta(G)-1}$ be a sequence of $\eta(G) - 1$ non-zero elements in G. Using the definition of $\eta(G)$, we may assume that S does not contain any non-empty subsequence S' such that $\sigma(S') = 0$, $|S'| \leq \exp(G)$. We put

$$T = a_1 \cdot a_2 \cdot \ldots \cdot a_{\eta(G)-1} \cdot \underbrace{0 \cdot \ldots \cdot 0}_{m\exp|G|-1\ \text{times}}.$$

We observe that sequence T does not contain m disjoint non-empty subsequences $T_1, T_2, \ldots, T_m$ such that $\sigma(T_i) = 0$ and $|T_i| \leq \exp(G)$ for $i \in [1, m]$. This implies that $\mathsf{s}_{(m)}(G) > \eta(G) + m\exp(G) - 2$. Hence $\mathsf{s}_{(m)}(G) \geq \eta(G) + m\exp(G) - 1$. On the other hand, if S is any sequence over G such that $|S| \geq \mathsf{s}(G) + (m-1)|G|$, then one can sequentially extract at least m disjoint subsequences $S_1, \ldots, S_m$, such that $\sigma(S_i) = 0$ in G and $|S_i| = \exp(G)$. Thus, $\mathsf{s}_{(m)}(G) \leq \mathsf{s}(G) + (m-1)\exp(G)$. ■

It was conjectured by Gao that for every finite Abelian group G, one has $\eta(G) + \exp(G) - 1 = \mathsf{s}(G)$, see (Gao & Geroldinger, 2006 [Conjecture 6.5]). If this conjecture is true, then by Theorem 4.4 for every finite Abelian group G the equality $\mathsf{s}_{(m)}(G) = \mathsf{s}(G) + (m-1)\exp(G)$ holds, i.e. the sequence $(\mathsf{s}_{(m)}(G))_{m\geq 1}$ is an arithmetic progression with difference $\exp(G)$. We will note that the equation $\eta(G) + \exp(G) - 1 = \mathsf{s}(G)$ is true for all finite Abelian groups of rank at most two, see (Girard & Schmid, 2019 [Theorem 2.3]). At this point it is worth mentioning that $\mathsf{s}(C_{n_1} \oplus C_{n_2}) = 2n_1 + 2n_2 - 3$ for all $1 < n_1 | n_2$, see (Girard & Schmid, 2019 [Theorem 2.3]).

Corollary 4.5. If $G = C_{n_1} \oplus C_{n_2}$, where $1 < n_1 | n_2$, then for a natural m we have:

$$\mathsf{E}_m(G) = mn_2n_1 + n_2 + n_1 - 2, \tag{5}$$

$$\mathsf{D}_m(G) = mn_2 + n_1 - 1, \tag{6}$$

$$\mathsf{s}_{(m)}(G) = (m+1)n_2 + 2n_1 - 3, \tag{7}$$

$$\mathsf{s}_{(m)}(G) - \mathsf{D}_m(G) = \mathsf{d}(G). \tag{8}$$

Proof. Equation (5) is a consequence of Remark 2.1 and Theorem 4.1; equation (6) follows from (Halter-Koch, 1992 [Proposition 5]). By applying (Girard & Schmid, 2019 [Theorem 2.3]) and Theorem 4.4, we can obtain (7). Equation (8) is a consequence of equations (6) and (7). Note that if G is an Abelian group, then $\mathsf{D}(G) - 1 = \mathsf{d}(G)$ is the maximal length of a zero-sum free sequence over G. ■

5. A generalization of the Kemnitz conjecture

Kemnitz's conjecture states that every set S of $4n-3$ lattice points in the plane has a subset S' with n points whose centroid is also a lattice point. This conjecture was proven by Christian Reiher (2007). In order to prove the generalization of this theorem, we will use equation (7).

Theorem 5.1. Let n and m be natural numbers. Let S be a set of $(m+3)n-3$ lattice points in 2-dimensional Euclidean space. Then there are at least m pairwise disjoint sets $S_1, S_2, \dots, S_m \subseteq S$ with n points each, such that the centroid of each set S_i is also a lattice point.

Proof. As Harborth has already noted see (Edel et al., 2007; Harborth, 1973), $\mathsf{s}(C_n^r)$ is the smallest integer l such that every set of l lattice points in r-dimensional Euclidean space contains n elements which have a centroid in a lattice point. It is known that $\mathsf{s}(C_n^2)=4n-3$ for all $n>1$, see (Girard & Schmid, 2019, [Theorem 2.3]). By analogy, $\mathsf{s}_{(m)}(C_n^r)$ is the smallest integer l such that every set S of l lattice points in r-dimensional Euclidean space has m pairwise disjoint subsets $S_1, S_2, \dots, S_m$ each of cardinality n, the centroids of which are also lattice points. Finally, in the case of 2-dimensional Euclidean space, it is sufficient to use equation (7), from which we get $\mathsf{s}_{(m)}(C_n^2)=(m+3)n-3$. The last equality completes the proof of Theorem 5.1. ■

Remark 5.2. As we can see in Definition 3.1, Theorem 5.1 is also true when we replace *sets* with *multisets*.

6. Some results on $\mathsf{s}_{I,m}(G)$ constant

In this section, we will investigate zero-sum constants for finite Abelian groups. We start with $\mathsf{s}_{\leq k}(G), \mathsf{D}_m(G), \eta(G)$ constants. Our main result of this section is Theorem 6.11. Olson (1969b) calculated $\mathsf{s}_{\leq p}(C_p^2)$ for a prime number p. No precise result is known for $\mathsf{s}_{\leq p}(C_p^n)$, where $n\geq 3$. We need two technical lemmas:

Lemma 6.1. Let p be a prime number and $n\geq 2$. Then:

$$\mathsf{s}_{\leq (n-1)p}(C_p^n) \leq (n+1)p-n. \tag{9}$$

Proof. Let $g_i \in C_p^n$, $i\in[1,(n+1)p-n]$. We embed group C_p^n into an Abelian group F which is isomorphic to C_p^{n+1}. Let $x\in F$, $x\notin C_p^n$. Since $\mathsf{D}(C_p^{n+1})=(n+1)p-n$ (see (Olson, 1969a) or Remark 2.1,(a)) there exists a zero-sum subsequence $\prod_{i\in I}(x+g_i)$ of the sequence $\prod_{i=1}^{(n+1)p-n}(x+g_i)$. But this is possible only if p divides $|I|$. Rearranging subscripts, we may assume that $g_1+g_2+\dots+g_{ep}=0$, where $e\in[1,n]$. The thesis is achieved if $e\in[1,n-1]$. If $e=n$ we obtain a zero-sum sequence $S=g_1\cdot g_2\cdot\ldots\cdot g_{np}$. Zero-sum sequence S contains a proper zero-sum subsequence S', since $\mathsf{D}(C_p^n)=np-(n-1)$, and thus a zero-sum subsequence of length not exceeding $\left[\frac{np+1}{2}\right]\leq(n-1)p$. ■

Corollary 6.2. Let p be a prime. Then:

$$\mathsf{s}_{\leq 2p}(C_p^3)\leq 4p-3\,. \tag{10}$$

Lemma 6.3. Let G be a finite Abelian group, $k\in\mathbb{N}$, $k\geq \exp(G)$. If $\mathsf{s}_{[1,k],1}(G)\leq \mathsf{s}_{[1,k],m}(G)+k$, then $\mathsf{s}_{[1,k],m+1}(G)\leq \mathsf{s}_{[1,k],m}(G)+k$.

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Proof. Let S be a sequence over G of length $\mathsf{s}_{[1,k],m}(G)+k$. The sequence S contains a non-empty subsequence $S_0|S$ such that $\sigma(S_0)=0, |S_0|\in[1,k]$, since $|S|\geq \mathsf{s}_{[1,k],1}(G)=\mathsf{s}_{\leq k}(G)$. By the definition of $\mathsf{s}_{[1,k],m}(G)$ the remaining elements in S contain m disjoint non-empty subsequences $S_i|S$ such that $\sigma(S_i)=0, |S_i|\in[1,k]$, where $i\in[1,m]$. Thus, we get $m+1$ non-empty disjoint subsequences $S_i|S$ such that $\sigma(S_i)=0, |S_i|\in[1,k]$, where $i\in[0,m]$. ■

Corollary 6.4. Let G be a finite Abelian group, $k\geq\exp(G)$. If $\mathsf{s}_{[1,k],1}(G)\leq \mathsf{s}_{[1,k],m}(G)+k$, then $\mathsf{s}_{[1,k],m+n}(G)\leq \mathsf{s}_{[1,k],m}(G)+nk$.

Proof. We use Lemma 6.3 and Remark 3.3. ■

Corollary 6.5. Let G be a finite Abelian group, $k\geq\exp(G)$. Then:

$$\mathsf{D}_n(G)\leq \mathsf{s}_{[1,k],n}(G)\leq \mathsf{s}_{\leq k}(G)+k(n-1), \tag{11}$$

$$\mathsf{D}_n(G)\leq \mathsf{s}_{[1,\exp(G)],n}(G)\leq \eta(G)+\exp(G)(n-1). \tag{12}$$

Proof. We use Remark 3.3, Corollary 6.4 with $m=1$ and get (11). We put $k=\exp(G)$ in (11) and get (12). ■

Remark 6.6. It is known that:

1. $\eta(C_n^3)=8n-7$, if $n=3^\alpha 5^\beta$, with $\alpha,\beta\geq 0$,
2. $\eta(C_n^3)=7n-6$, if $n=3\cdot 2^\alpha$, with $\alpha\geq 1$,
3. $\eta(C_2^3)=8$, $\eta(C_3^3)=17$, $\eta(C_3^4)=39$, $\eta(C_3^5)=89$, $\eta(C_3^6)=223$ (Girard, 2018).

Corollary 6.7. We have:

1. $\mathsf{D}_m(C_n^3)\leq nm+7n-7$, if $n=3^\alpha 5^\beta$, with $\alpha,\beta\geq 0$,
2. $\mathsf{D}_m(C_n^3)\leq nm+6n-6$, if $n=3\cdot 2^\alpha$, with $\alpha\geq 1$,
3. $\mathsf{D}_m(C_3^4)\leq 3m+36$, $\mathsf{D}_m(C_3^5)\leq 3m+86$, $\mathsf{D}_m(C_3^6)\leq 3m+220$.

Proof. By Corollary 6.5 and Remark 6.6. ■

In the next Lemma, we collect several useful properties on the Davenport constant.

Lemma 6.8. Let G be a non-trivial finite Abelian group and H be a subgroup of G. Then:

$$\mathsf{D}(H)+\mathsf{D}(G/H)-1\leq \mathsf{D}(G)\leq \mathsf{D}_{\mathsf{D}(H)}(G/H)\leq \mathsf{D}(H)\mathsf{D}(G/H). \tag{13}$$

Proof. The inequality $\mathsf{D}(H)+\mathsf{D}(G/H)-1\leq\mathsf{D}(G)$ is proven in (Halter-Koch, 1992 [Proposition 3 (i)]). Now we prove the inequality $\mathsf{D}(G)\leq\mathsf{D}_{\mathsf{D}(H)}(G/H)$ on the same lines as in (Delorme, Ordaz & Quiroz, 2001), we include the proof for the sake of completeness.

If $|S|\geq\mathsf{D}_{\mathsf{D}(H)}(G/H)$ is any sequence over G, then one can, by definition, extract at least $\mathsf{D}(H)$ disjoint non-empty subsequences $S_1,\ldots,S_{\mathsf{D}(H)}|S$ such that $\sigma(S_i)\in H$. Since $T=\prod_{i=1}^{D(H)}\sigma(S_i)$ is a sequence over H of length $\mathsf{D}(H)$, there thus exists a non-empty subset $I\subseteq[1,\mathsf{D}(H)]$ such that $T'=\prod_{i\in I}\sigma(S_i)$ is a zero-sum subsequent of T.

We obtain that $S'=\prod_{i\in I}S_i$ is a non-empty zero-sum subsequence of S.

The inequality $\mathsf{D}_{\mathsf{D}(H)}(G/H)\leq\mathsf{D}(H)\mathsf{D}(G/H)$ follows from Remark 3.3. ■

Theorem 6.9. For an Abelian group $C_p \oplus C_{n_2} \oplus C_{n_3}$ such that $p|n_2|n_3 \in \mathbb{N}$, where p is a prime number, we have

$$n_3+n_2+p-2 \leq \mathsf{D}\left(C_p \oplus C_{n_2} \oplus C_{n_3}\right) \leq \mathsf{D}_{\frac{n_2}{p}+\frac{n_3}{p}-1}\left(C_p^3\right) \leq 2n_3+2n_2-3. \tag{14}$$

Proof. If $G=C_p \oplus C_{n_2} \oplus C_{n_3}$ such that $p|n_2|n_3 \in \mathbb{N}$, then $\exp(G)=n_3$. Note that $n_3+n_2+p-2 = \mathsf{D}^*(G) \leq \mathsf{D}(G)$. Denoting by H a subgroup of G such that $H \cong C_{\frac{n_2}{p}} \oplus C_{\frac{n_3}{p}}$. The quotient group $G/H \cong C_p \oplus C_p \oplus C_p$. By Lemma 6.8 we get

$$\mathsf{D}(G) \leq \mathsf{D}_{\mathsf{D}(H)}(G/H) = \mathsf{D}_{\frac{n_2}{p}+\frac{n_3}{p}-1}\left(C_p^3\right). \tag{15}$$

By (11) (with $m=\frac{n_2}{p}+\frac{n_3}{p}-1, k=2p$) and Corollary 6.2, we get

$$\mathsf{D}(G) \leq \mathsf{s}_{\leq 2p}\left(C_p^3\right)+2p\left(\frac{n_2}{p}+\frac{n_3}{p}-2\right) \leq 2p\left(\frac{n_2}{p}+\frac{n_3}{p}-2\right)+4p-3= \tag{16}$$
$$=2n_3+2n_2-3.$$

∎

Our next goal is to generalize (Delorme, Ordaz & Quiroz, 2001 [Theorem 3.2]) to the case $r(G) \geq 3$.

Theorem 6.10. Let H, K and L be Abelian groups of orders $|H|=h$, $|K|=k$ and $|L|=l$. If $G=H \oplus K \oplus L$ with $h|k|l$. Let $\Omega(h)$ denote the total number of prime factors of h. Then:

$$\mathsf{s}_{\leq 2^{\Omega(h)}l}(G) \leq 2^{\Omega(h)}(2l+k+h)-3. \tag{17}$$

Proof. The proof will be inductive. If $h=1$, then by (Delorme, Ordaz & Quiroz, 2001 [Theorem 3.2]) we have

$$\mathsf{s}_{\leq 2^{\Omega(h)}l}(G) = \mathsf{s}_{\leq l}(K \oplus L) \leq 2l+k-2 = 2^{\Omega(1)}(2l+k+1)-3. \tag{18}$$

Assume that $h>1$ and let p be a prime divisor of h.
Let H_1 be a subgroup of H, K_1 be a subgroup of K, L_1 be a subgroup of L, with indices $[H:H_1]=[K:K_1]=[L:L_1]=p$. Put $h=ph_1$, $k=pk_1$, $l=pl_1$ and $Q=H_1 \oplus K_1 \oplus L_1$. Assume inductively that theorem is true for Q i.e.

$$\mathsf{s}_{\leq 2^{\Omega(h_1)}l_1}(Q) \leq 2^{\Omega(h_1)}(2l_1+k_1+h_1)-3. \tag{19}$$

Let $s=2^{\Omega(h)}(2l+k+h)-3$ and $S=g_1 \cdot g_2 \cdot \ldots \cdot g_s$ be a sequence of G.
We shall prove that there exists a subsequence of S with a length smaller than or equal to $2^{\Omega(h)}l$ and a zero sum. Let $b_i=g_i+Q \in G/Q$. We consider the sequence $\prod_{i=1}^{s} b_i$ of length s. The quotient group G/Q is isomorphic to C_p^3 and

$$s=2p(2^{\Omega(h_1)}(2l_1+k_1+h_1)-2)+4p-3. \tag{20}$$

Therefore, by Corollary 6.2 there exists at least j_0 pairwise disjoint sets $I_j \subseteq [1,s]$, $|I_j| \leq 2p$, where

$$j \leq j_0 = 2^{\Omega(h_1)}(2l_1+k_1+h_1)-1, \tag{21}$$

such that each sequence $\prod_{i \in I_j} b_i$ has a zero sum in G/Q. In other words $\sigma\left(\prod_{i \in I_j} g_i\right)=\sum_{i \in I_j} g_i \in Q$. By induction assumption for Q there exists $J \subseteq [1,j_0]$ with $|J| \leq 2^{\Omega(h_1)}l_1$ such that $\sum_{j \in J} \sigma\left(\prod_{i \in I_j} g_i\right)=0$. Thus, we obtain a zero-sum

https://doi.org/10.37705/TechTrans/e2020027

subsequence of S in G of length not exceeding $\sum_{j\in J}|I_j|\le 2^{\Omega(h_1)}l_1\cdot 2p=2^{\Omega(h)}l$, which ends the inductive proof.

Theorem 6.11. Let $H_1, H_2, ..., H_n$ be Abelian groups of orders $|H_i|=h_i$. If $n\ge 2$ and $G=H_1\oplus H_2\oplus \dots \oplus H_n$ with $h_1|h_2|...|h_n$, then

$$\mathsf{s}_{\le(n-1)^{\Omega(h_n)}h_n}(G)\le(n-1)^{\Omega(h_n)}(2(h_n-1)+(h_{n-1}-1)+\dots+(h_1-1)+1). \tag{22}$$

Proof. We proceed by induction on n and h_n.
If $n=2$, then the inequality (22) holds by (Delorme, Ordaz & Quiroz, 2001 [Theorem 3.2]). Namely

$$\begin{aligned}\mathsf{s}_{\le(2-1)^{\Omega(h_2)}h_2}(H_1\oplus H_2)=\mathsf{s}_{\le h_2}(H_1\oplus H_2)\le 2h_2+h_1-2=\\ =(2-1)^{\Omega(h_2)}(2(h_2-1)+(h_1-1)+1).\end{aligned} \tag{23}$$

Suppose that the inequality (22) holds for fixed $n-1\ge 2$:

$$\begin{aligned}\mathsf{s}_{\le(n-2)^{\Omega(h_{n-1})}h_{n-1}}(H_1\oplus\dots\oplus H_{n-1})\le\\ \le(n-2)^{\Omega(h_{n-1})}(2(h_{n-1}-1)+(h_{n-2}-1)+\dots+(h_1-1)+1).\end{aligned} \tag{24}$$

If $n\ge 3$ and $h_1=1$, then G and $H_2\oplus\dots\oplus H_n$ are isomorphic. By (2):

$$\begin{aligned}\mathsf{s}_{\le(n-1)^{\Omega(h_n)}h_n}(G)=\mathsf{s}_{\le(n-1)^{\Omega(h_n)}h_n}(H_2\oplus\dots\oplus H_n)\le\\ \le\mathsf{s}_{\le(n-2)^{\Omega(h_n)}h_n}(H_2\oplus\dots\oplus H_n).\end{aligned} \tag{25}$$

Thus by the induction hypothesis (24):

$$\begin{aligned}\mathsf{s}_{\le(n-1)^{\Omega(h_n)}h_n}(G)\le\\ \le(n-2)^{\Omega(h_n)}(2(h_n-1)+(h_{n-1}-1)+\dots+(h_2-1)+1)\le\\ \le(n-1)^{\Omega(h_n)}(2(h_n-1)+(h_{n-1}-1)+\dots+(h_2-1)+(1-1)+1).\end{aligned} \tag{26}$$

Therefore (22) holds.
Suppose that the inequality (22) holds for fixed $n\ge 3$ and fixed h, such that $h_1>h\ge 1$:

$$\begin{aligned}\mathsf{s}_{\le(n-1)^{\Omega(h_n)}h_n}(G)\le\\ \le(n-1)^{\Omega(h_n)}(2(h_n-1)+(h_{n-1}-1)+\dots+(h-1)+1).\end{aligned} \tag{27}$$

Let p be a prime divisor of h_1. Let H_i^* be a subgroup of index p of a group H_i. Put $h_i=ph_i^*$ and $Q=H_1^*\oplus H_2^*\oplus\dots\oplus H_n^*$. By inductive assumption, the inequality (22) holds for Q:

$$\mathsf{s}_{\le(n-1)^{\Omega(h_n^*)}h_n^*}(Q)\le 2^{\Omega(h_n^*)}(2(h_n^*-1)+(h_{n-1}^*-1)+\dots+(h_1^*-1)+1). \tag{28}$$

We put $s=(n-1)^{\Omega(h_n)}(2(h_n-1)+(h_{n-1}-1)+\dots+(h_1-1)+1)$ and let $S=g_1\cdot g_2\cdot\ldots\cdot g_s$ be a sequence of G.
We shall prove that there exists a subsequence of S with a length smaller than or equal to $(n-1)^{\Omega(h_n)}h_n$ and a zero sum. Let $b_i=g_i+Q, 1\le i\le s$, be the sequence of G/Q.
The quotient group G/Q is isomorphic to C_p^n and

$$\begin{aligned}s=(n-1)^{\Omega(h_n^*)+1}(p(2(h_n^*-1)+\dots+(h_1^*-1)+1)+n(p-1))\ge\\ \ge p(n-1)^{\Omega(h_n^*)+1}(2(h_n^*-1)+\dots+(h_1^*-1)+1)+(n-1)n(p-1)\ge\\ \ge(n-1)p((n-1)^{\Omega(h_n^*)}(2(h_n^*-1)+\dots+(h_1^*-1)+1))+2p-n.\end{aligned} \tag{29}$$

Therefore, by Lemma 6.1 there exists at least j_0 pairwise disjoint sets $I_j\subseteq[1,s]$ with $|I_j|\le(n-1)p$ and

$$j\le j_0=(n-1)^{\Omega(h_n^*)}(2(h_n^*-1)+\dots+(h_1^*-1)+1), \tag{30}$$

such that sequence $\prod_{i\in I_j} b_i$ has a zero sum in G/Q.

In another words $\sigma\left(\prod_{i\in I_j} g_i\right)=\sum_{i\in I_j} g_i \in Q$. By induction assumption (22) for Q there exists $J\subseteq[1,j_0]$ with $|J|\leq(n-1)^{\Omega(h_n^*)}h_n^*$ such that $\sum_{j\in J}\sigma\left(\prod_{i\in I_j} g_i\right)=0$. Thus, we obtain a zero sum subsequence of S in G of length not exceeding $\sum_{j\in J}\left|I_j\right|\leq(n-1)^{\Omega(h_n^*)}h_n^*(n-1)p=(n-1)^{\Omega(h_n)}h_n$. ■

7. The smooth numbers

First, we recall the notation of a smooth number. Let $F=\{q_1,q_2,\dots,q_r\}$ be a subset of positive integers. A positive integer k is said to be smooth over a set F if $k=q_1^{e_1}\cdot q_2^{e_2}\cdot\ldots\cdot q_r^{e_r}$, where e_i are non-negative integers.

Remark 7.1. Let $n\in\mathbb{N}$. Each smooth number over a set $\{q_1^n,q_2^n,\dots,q_r^n\}$ is an n-th power of a suitable smooth number over the set $\{q_1,q_2,\dots,q_r\}$.

Definition 7.2. Let $\{p_1,p_2,\dots,p_r\}$ be a set of distinct prime numbers. By $\mathsf{c}(n_1,n_2,\dots,n_r)$, we denote the smallest $t\in\mathbb{N}$ such that every sequence M of smooth numbers over a set $\{p_1,p_2,\dots,p_r\}$, of length t has a non-empty subsequence N such that the product of all the terms of N is a smooth number over a set $\left\{p_1^{n_1},\ p_2^{n_2},\ \dots,\ p_r^{n_r}\right\}$.

In the next theorem, we use notation $\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\ldots\oplus\mathbb{Z}_{n_r}$ instead of notation $C_{n_1}\oplus C_{n_2}\oplus\ldots\oplus C_{n_r}$, these two structures are isomorphic to one another.

Theorem 7.3. Let $n_1,n_2,\dots,n_r$ be integers such that $1<n_1|n_2|\dots|n_r$. Then:

$$\mathsf{c}(n_1,n_2,\dots,n_r)=\mathsf{D}(\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\ldots\oplus\mathbb{Z}_{n_r}). \tag{31}$$

Proof. It follows on the same lines as the proof of (Chintamani et al., 2012 [Theorem 1.6.]). First, we will prove that

$$\mathsf{c}(n_1,n_2,\dots,n_r)\leq\mathsf{D}(\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\ldots\oplus\mathbb{Z}_{n_r}). \tag{32}$$

We put $l=\mathsf{D}(\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\ldots\oplus\mathbb{Z}_{n_r})$. Let $M=(m_1,m_2,\dots,m_l)$ be a sequence of smooth numbers with respect to $F=\{p_1,p_2,\dots,p_r\}$.

For all $i\in[1,l]$, we have $m_i=p_1^{e_{i,1}}\cdot p_2^{e_{i,2}}\cdot\ldots\cdot p_r^{e_{i,r}}$, where $e_{i,j}$ are non-negative integers.

We associate each m_i with $a_i\in\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\ldots\oplus\mathbb{Z}_{n_r}$ under the homomorphism:

$$\Phi:\left\{p_1^{e_1}\cdot p_2^{e_2}\cdot\ldots\cdot p_r^{e_r}:e_i\geq0,e_i\in\mathbb{Z}\right\}\to\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\ldots\oplus\mathbb{Z}_{n_r}, \tag{33}$$

$$\Phi\left(p_1^{e_1}\cdot p_2^{e_2}\cdot\ldots\cdot p_r^{e_r}\right)=\left(\left[e_1\right]_{n_1},\left[e_2\right]_{n_2},\dots,\left[e_r\right]_{n_r}\right).$$

Hence

$$\Phi(m_i)=\left(\left[e_{i,1}\right]_{n_1},\left[e_{i,2}\right]_{n_2},\dots,\left[e_{i,r}\right]_{n_r}\right), \tag{34}$$

where $i\in[1,l]$.

Thus, we get a sequence $S=a_1a_2\cdot\ldots\cdot a_l$ of elements of the group $\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\ldots\oplus\mathbb{Z}_{n_r}$ of length $l=\mathsf{D}(\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\ldots\oplus\mathbb{Z}_{n_r})$. Therefore, there exists a non-empty zero sum subsequence T of S in $\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\ldots\oplus\mathbb{Z}_{n_r}$. Let $T=a_{j_1}a_{j_2}\cdot\ldots\cdot a_{j_t}$, where $a_{j_i}=\left(\left[e_{j_i,1}\right]_{n_1},\left[e_{j_i,2}\right]_{n_2},\dots,\left[e_{j_i,r}\right]_{n_r}\right)$ and

$$\sum_{i=1}^{t}e_{j_i,k}\equiv0\left(\mathrm{mod}\,n_k\right), \tag{35}$$

where $k\in[1,r]$.

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Consider the subsequence N of M corresponding to T. We have $N=(m_{j_1}, m_{j_2}, \dots, m_{j_t})$ and by equation (35), we get

$$\prod_{i=1}^{t} m_{j_i} = \prod_{k=1}^{r} p_k^{\sum_{i=1}^{t} e_{j_i,k}} = \prod_{k=1}^{r} \left(p_k^{n_k}\right)^{l_k}, \tag{36}$$

for some integers $l_k \geq 0$. Thus, the product $\prod_{i=1}^{t} m_{j_i}$ of all the terms of N is a smooth number over a set $\left\{p_1^{n_1}, p_2^{n_2}, \dots, p_r^{n_r}\right\}$. By definition of $\mathsf{c}(n_1, n_2, \dots, n_r)$ we get inequality (32).

On the other hand, we will now prove that

$$\mathsf{c}(n_1, n_2, \dots, n_r) \geq \mathsf{D}(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_r}). \tag{37}$$

Let $l=\mathsf{c}(n_1, n_2, \dots, n_r)$ and $S=a_1 a_2 \cdot \dots \cdot a_l$ be a sequence of elements of $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_r}$ of length l, where $a_i = \left(\left[e_{i,1}\right]_{n_1}, \left[e_{i,2}\right]_{n_2}, \dots, \left[e_{i,r}\right]_{n_r}\right)$.

We put $m_i = p_1^{e_{i,1}} \cdot p_2^{e_{i,2}} \cdot \dots \cdot p_r^{e_{i,r}}$. The sequence $M=(m_1, m_2, \dots, m_l)$ of integers is a sequence of smooth numbers over a set F.

By definition of $l=\mathsf{c}(n_1, n_2, \dots, n_r)$, there exists a non-empty subsequence $N=(m_{j_1}, m_{j_2}, \dots, m_{j_t})$ of M such that

$$\prod_{i=1}^{t} m_{j_i} = \prod_{k=1}^{r} \left(p_k^{n_k}\right)^{l_k}, \tag{38}$$

For some integers $l_k \geq 0$. The subsequence T of S corresponding to N will sum up to the identity in $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \dots \oplus \mathbb{Z}_{n_r}$. Therefore, (37) holds and we obtain (31). ∎

8. Future work and the non-Abelian case

The constant $\mathsf{E}(G)$ has received a lot of attention over the last ten years. For example, one direction went towards weighted zero-sum problems, see (Grynkiewich, 2013 [Chapter 16]). The second direction went towards non-Abelian groups.

Let G be any additive finite group. Let $S=(a_1, a_2, \dots, a_n)$ be a sequence over G. We say that the sequence S is a zero-sum sequence if there exists a permutation $\sigma:[1,n]\longrightarrow[1,n]$ such that $0=a_{\sigma(1)}+\dots+a_{\sigma(n)}$. For a subset $I \subseteq \mathbb{N}$, let $\mathsf{s}_I(G)$ denote the smallest $t \in \mathbb{N} \cup \{0, \infty\}$ such that every sequence S over G of length $|S| \geq t$ has a zero-sum subsequence S' of length $|S'| \in I$. The constants $\mathsf{D}(G) := \mathsf{s}_{\mathbb{N}}(G)$ and $\mathsf{E}(G) := \mathsf{s}_{\{|G|\}}(G)$ are classical invariants in zero-sum theory (independently of whether G is Abelian or not). We recall that for a given finite group G, we denote by $\mathsf{d}(G)$, the maximal length of a zero-sum free sequence over G. We call $\mathsf{d}(G)$ the small Davenport constant.

Remark 8.1. In the Abelian case $\mathsf{d}(G)$=$\mathsf{D}(G)$–1. Note that in the case of non-Abelian groups, $\mathsf{d}(G)$ can be strictly smaller than $\mathsf{D}(G)-1$, see (Geroldinger & Ruzsa, 2009 [Chapter 2]).

Theorem 8.2. If G is a finite group of order $|G|$, then

$$\mathsf{d}(G)+m|G| \leq \mathsf{E}_m(G) \leq \mathsf{E}(G)+(m-1)|G|. \tag{39}$$

Proof. Let $S=(a_1, a_2, \dots, a_{\mathsf{d}(G)})$ be a sequence of $\mathsf{d}(G)$ non-zero elements in G. Using the definition of $\mathsf{d}(G)$, we may assume that S does not contain any non-empty subsequence S' such that $\sigma(S')=0$. We put

$$T=(a_1, a_2, \dots, a_{\mathsf{d}(G)}, 0, \dots, 0), \tag{40}$$

where $v_0(T)=m|G|-1$.

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We observe that the sequence T does not contain m disjoint non-empty subsequences $T_1, T_2, \ldots, T_m$ of T such that $\sigma(T_i)=0$ and $|T_i|=|G|$ for $i \in [1,m]$. This implies that $\mathsf{E}_m(G) > \mathsf{d}(G)+m|G|-1$. Hence

$$\mathsf{E}_m(G) \geq \mathsf{d}(G)+m|G|. \tag{41}$$

On the other hand, if S is any sequence over G such that

$$|S| \geq \mathsf{E}(G)+(m-1)|G|,$$

then one can sequentially extract at least m disjoint subsequences $S_1, \ldots, S_m$ of S, such that $\sigma(S_i)=0$ in G and $|S_i|=|G|$. Thus

$$\mathsf{E}_m(G) \leq \mathsf{E}(G)+(m-1)|G|. \tag{42}$$ ■

Corollary 8.3. If $m \to \infty$, then $\mathsf{E}_m(G) \sim m|G|$.
Proof. If $m \to \infty$, then $\frac{\mathsf{E}_m(G)}{m} \to |G|$ by the inequality (39).

■

We now give an application of Theorem 8.2. The formula $\mathsf{E}(G)=\mathsf{d}(G)+|G|$ was proved for all finite Abelian groups and for some classes of finite non-Abelian groups (see equation (3) and (Bass, 2007; Han, 2015; Han & Zhang, 2019; Oh & Zhong, 2019)). Thus

$$\mathsf{E}_m(G)=\mathsf{d}(G)+m|G|$$

holds for finite groups in the following classes: Abelian groups, nilpotent groups, groups in the form $C_m \ltimes_{\phi} C_{mn}$, where $m, n \in \mathbb{N}$, dihedral and dicyclic groups and all non-Abelian groups of order pq with p and q prime. Therefore, the following conjecture can be proposed:

Corollary 8.4. (See (Bass, 2007 [Conjecture 2]))
For any finite group G, we have the equation $\mathsf{E}_m(G)=\mathsf{d}(G)+m|G|$.

9. Conclusion

This paper makes a contribution to the theory of additive combinatorics. It provides an overview of the state of knowledge of the zero-sum problems and can be considered as an introduction to this theory. We have proven that if G is an Abelian group, then $\mathsf{E}_m(G)=\mathsf{d}(G)+m|G|$. We have studied the asymptotic behaviour of the sequences $(\mathsf{E}_m(G))_{m\geq 1}$ and $(\eta_m(G))_{m\geq 1}$. For a prime p and a natural $n \geq 2$, we have derived the inequality $\mathsf{s}_{\leq(n-1)p}(C_p^n) \leq (n+1)p-n$. We have proven a generalization of Kemnitz's conjecture. We have applied the Davenport constant to smooth numbers. Finally, we have shown some results in the non-Abelian case.

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