The zero-sum constant, the Davenport constant and their analogues

Maciej Zakarczemny
mzakarczemny@pk.edu.pl | http://orcid.org/0000-0002-7202-1244
Department of Applied Mathematics, Faculty of Computer Science and Telecommunications, Cracow University of Technology

Abstract

Let $D(G)$ be the Davenport constant of a finite Abelian group $G$. For a positive integer $m$ (the case $m=1$, is the classical case) let $E_m(G)$ (or $\eta_m(G)$) be the least positive integer $t$ such that every sequence of length $t$ in $G$ contains $m$ disjoint zero-sum sequences, each of length $|G|$ (or of length $\exp(G)$, respectively). In this paper, we prove that if $G$ is an Abelian group, then $E_m(G)=D(G)-1+m|G|$, which generalizes Gao's relation. Moreover, we examine the asymptotic behaviour of the sequences $(E_m(G))_{m \geq 1}$ and $(\eta_m(G))_{m \geq 1}$. We prove a generalization of Kemnitz's conjecture. The paper also contains a result of independent interest, which is a stronger version of a result by Ch. Delorme, O. Ordaz, D. Quiroz. At the end, we apply the Davenport constant to smooth numbers and make a natural conjecture in the non-Abelian case.

Keywords: zero-sum sequence, Davenport constant, finite Abelian group
1. Introduction

We will define and investigate some generalizations of the Davenport constant, see (Alon & Dubiner, 1995; Edel et al., 2007; Freeze & Schmid, 2010; Gao & Geroldinger, 2006; Geroldinger & Halter-Koch, 2006; Olson, 1969a, 1969b; Reiner, 2007; Rogers, 1963). Davenport’s constant is connected with algebraic number theory as follows. For an algebraic number field $\mathbb{K}$, let $\mathcal{O}_\mathbb{K}$ be its ring of integers and $G$ the ideal class group of $\mathcal{O}_\mathbb{K}$. Let $x \in \mathcal{O}_\mathbb{K}$ be an irreducible element. If $\mathcal{O}_\mathbb{K}$ is a Dedekind domain, then $\mathcal{O}_\mathbb{K} = \prod P_i$, where $P_i$ are prime ideals in $\mathcal{O}_\mathbb{K}$ (not necessarily distinct). The Davenport constant $D(G)$ is the maximal number of prime ideals $P_i$ (counted with multiplicities) in the prime ideal decomposition of the integral ideal $x\mathcal{O}_\mathbb{K}$, see (Halter-Koch, 1992; Olson, 1969a).

The precise value of the Davenport constant is known, among others, for $p$-groups and for groups of rank at most two. The determination of $D(G)$ for general finite Abelian groups is an open question, see (Girard, 2018).

2. General notation

Let $\mathbb{N}$ denote the set of positive integers (natural numbers).

We set $[a, b] = \{x : a \leq x \leq b, x \in \mathbb{Z}\}$, where $a, b \in \mathbb{Z}$. Our notation and terminology is consistent with (Geroldinger & Ruzsa, 2009). Let $G$ be a non-trivial additive finite Abelian group. $G$ can be uniquely decomposed as a direct sum of cyclic groups $C_{n_1} \oplus C_{n_2} \oplus \ldots \oplus C_{n_r}$ with natural numbers $1 < n_1, n_2, \ldots, n_r$. The number $r$ of summands in the above decomposition of $G$ is expressed as $r = r(G)$ and called the rank of $G$. The integer $n_i$ is called the exponent of $G$ and denoted by $\exp(G)$.

In addition, we define $D'(G)$ as $D'(G) = 1 + \sum_{n_i} (n_i - 1)$. We write any finite sequence $S$ of $l$ elements of $G$ in the form $\prod_{s \in S} g^{v_s(s)} = g_1 \cdot g_2 \cdot \ldots g_l$ (this is a formal Abelian product), where $l$ is the length of $S$ denoted by $|S|$, and $v_s(S)$ is the multiplicity of $g$ in $S$. $S$ corresponds to the sequence (in the traditional sense) $(g_1, g_2, \ldots, g_l)$, where we forget the ordering of the terms. By $\sigma(S)$ we denote the sum of $S : \sigma(S) = \sum_{s \in S} v_s(S)g \in G$.

The Davenport constant $D(G)$ is defined as the smallest $t \in \mathbb{N}$ such that each sequence over $G$ of length at least $t$ has a non-empty zero-sum subsequence. Equivalently, $D(G)$ is the maximal length of a zero-sum sequence of elements of $G$ and with no proper zero-sum subsequence. One of the best bounds for $D(G)$ known so far is:

$$D'(G) \leq D(G) \leq n_1 \left(1 + \log \frac{|G|}{n_1}\right). \tag{1}$$

The small Davenport constant $d(G)$ is the maximal length of a zero-sum free sequence over $|G|$. If $|G|$ is a finite Abelian group then $d(G) = D(G) - 1$, see (Geroldinger & Ruzsa, 2009 [Definition 2.1.1.]). Alford, Granville and Pomerance (1994) used the bound (1) to prove the existence of infinitely many Carmichael numbers. Dimitrov (2007) used the Alon Dubiner constant (Alon & Dubiner, 1995) to prove the inequality:

$$\frac{D(G)}{D'(G)} \leq (Kr \log r)^t,$$

for an absolute constant $K$. It is known that for groups of rank at most two and for $p$-groups, where $p$ is a prime, the left hand side inequality (1) is in fact an equality, see (Olson, 1969a, 1969b). This result suggests that $D'(G) = D(G)$. However, there are infinitely many groups $G$ with rank $r > 3$ such that $D(G) > D'(G)$. There are more recent results on groups where the Davenport constant does not match the usual
lower bound, see (Geroldinger & Schneider, 1992). The following Remark 2.1 lists some basic facts for the Davenport constant, see (Delorme, Ordaz & Quiroz, 2001; Geroldinger & Schneider, 1992; Schmid, 2011; Sheikh, 2017).

Remark 2.1. Let $G$ be a finite additive Abelian group.
1. Then $D(G) = D'(G)$ in each of the following cases:
   - $G$ is a $p$-group;
   - $G$ has rank $r \leq 2$;
   - $G = C_p \oplus C_p \oplus C_{p^m}$ with $p$ a prime number, $n \geq 2$ and $m$ a natural number coprime with $p^r$ (more generally, if $G = G_1 \oplus C_{p^m}$ where $G_1$ is a $p$-group and $p^r \geq D(G_1)$);
   - $G = C_2 \oplus C_2 \oplus C_2 \oplus C_{2n}$, with odd $n$;
   - $G = C_{2k_1} \oplus C_{2k_2} \oplus C_{2k_3}$, with $p$ prime, $k_1, k_2, k_3 \geq 0$;
   - $G = C_2 \oplus C_2 \oplus C_{2n}$, with $n, m$ natural numbers;
   - $G$ has one of the forms $C_2 \oplus C_2 \oplus C_{3n}$, $C_2 \oplus C_2 \oplus C_{4n}$, or $C_2 \oplus C_2 \oplus C_{6n}$, with $n$ a natural number;
   - $G = C_4 \oplus C_4 \oplus C_{10}$.

2. Then $D(G) > D'(G)$ in each of the following cases:
   - $G = C_2 \oplus C_2 \oplus C_2 \oplus C_{2n}$, with odd $n \geq 3$;
   - $G = C_{2k} \oplus C_{2n}$, with $k \geq 3$ and odd $n \geq 3$;
   - $G = C_2 \oplus C_2 \oplus C_2 \oplus C_2$;
   - $G = C_2 \oplus C_2 \oplus C_2 \oplus C_{15}$;
   - let $n \geq 2$, $k \geq 2$, $(n, k) = 1$, $0 \leq \rho \leq n - 1$, and $G = C_{(k-1)n+\rho} \oplus C_{4n}$.
     - If $\rho \geq 1$ and $\rho \equiv n \pmod{k}$, then $D(G) \geq D'(G) + \rho$.
     - If $\rho \leq n - 2$ and $x(n-1-\rho) \not\equiv n \pmod{k}$ for any $x \in [1, n-1]$, then $D(G) \geq D'(G) + \rho + 1$.

3. Definitions

In this section, we will provide some definitions of classical invariants. We begin with some notations and remarks that will be used throughout the paper.

Definition 3.1. Let $G$ be a finite Abelian group, and $m, k$ be positive integers such that $k \geq \exp(G)$, and $\mathcal{O} \subseteq \mathbb{N}$.
1. By $s_0(G)$ we denote the smallest $t \in \mathbb{N} \cup \{\infty\}$ such that every sequence $S$ (with repetition allowed) over $G$ of length $t$ contains a non-empty subsequence $S'$ such that $\sigma(S') = 0$, $|S'| \in I$. We use notation $s_m(G)$ or $D'(G)$ to denote $s(G)$ if $I = [1, k]$, see (Balasubramanian & Bhowmik, 2006; Chintamani et al., 2012; Delorme, Ordaz, & Quiroz, 2001).
2. By $s_m(G)$ we denote the smallest $t \in \mathbb{N} \cup \{\infty\}$ such that every sequence $S$ over $G$ of length $t$ contains at least $m$ disjoint non-empty subsequences $S_1, S_2, ..., S_m$ such that $\sigma(S) = 0$, $|S| \in I$.
3. Let $E(G) = s_{[1|\infty]}(G)$, i.e. the smallest $t \in \mathbb{N} \cup \{\infty\}$ such that every sequence $S$ over $G$ of length $t$ contains non-empty subsequence $S'$ such that $\sigma(S') = 0$, $|S'| = |G|$. Note that $E(G)$ is the classical zero-sum constant.
4. Let $E_m(G) = s_{[1|\infty]}(G)$, i.e. the smallest $t \in \mathbb{N} \cup \{\infty\}$ such that every sequence $S$ over $G$ of length $t$ contains at least $m$ non-empty subsequences $S_1, S_2, ..., S_m$ such that $\sigma(S) = 0$, $|S| = |G|$.
5. Also, we define $\eta(G) = s_{[1|\infty]}(G)$, $s(G) = s_{[1|\infty]}(G)$, $D_m(G) = s_{m,m}(G)$ (see (Halter-Koch, 1992)), $s_{\infty}(G) = s_{[1|\infty]}(G)$.

Remark 3.2. For $n, n' \in \mathbb{N}$, $\mathcal{O} \subseteq \mathbb{N}$, by definition $s_n(G) = s_{[1|\infty]}(G) = D(G) = D_n(G)$, $s_{n+1}(G) = s_n(G)$, $s_{[1|\infty]}(G) = D_n(G)$, $s_{[1|\infty]}(G) = E(G)$.
Note that $s(G) := s_{\text{exp}(G)}(G)$ is the classical Erdős-Ginzburg-Ziv constant (Fan, Gao, & Zhong, 2011). We call $s_m(G) = s_{\text{exp}(G),m}(G)$ the $m$-wise Erdős-Ginzburg-Ziv constant of $G$. Thus, in this notation $s_1(G) = s(G)$.

One can derive, for example, the following inequalities.

**Remark 3.3.** If $n' \geq n$, then $s_{m+1}(G) \geq s_m(G)$. If $D(G) \geq k \geq \exp(G)$, then $s_{\exp(G),m}(G) \geq s_{\exp(G),km}(G) \geq D_\exp(G)$. If $D(G) \geq k \geq \exp(G)$, then $\eta(G) \geq s_{\Delta}(G) \geq D(G)$. If $k \geq k'$, then

$$s_{\Delta}(G) \geq s_{\Delta}(G).$$

We note that a sequence $S$ over $G$ of length $|S| \geq mD(G)$ can be partitioned into $m$ disjoint subsequences $S_i$ of length $|S_i| \geq D(G)$. Thus, each $S_i$ contains a non-empty zero-sum subsequence. Hence $D_{\Delta}(G) \leq mD(G)$. See also (Halter-Koch, 1992 [Proposition 1 (ii)])

**4. The $m$-wise zero-sum constant and the $m$-wise Erdős-Ginzburg-Ziv constant of $G$**

In 1996, Gao and Caro independently proved that

$$E(G) = D(G) + |G| - 1$$

for any finite Abelian group, see (Caro, 1996; Gao, 1995, 1996). For a proof in modern language we refer to (Geroldinger & Halter-Koch, 2006 [Proposition 5.7.9]); see also (Delorme, Ordaz & Quiroz, 2001; Gao, 1994; Hamidoune, 1996). Relation (3) unifies research on constants $D(G)$ and $E(G)$. We start this section with the result that can be used to unify research on constants $E(G)$ and $E_{\Delta}(G)$.

**Theorem 4.1.** If $G$ is a finite Abelian group of order $|G|$, then

$$E_{\Delta}(G) = E(G) + (m - 1)|G| = D(G) - 1 + m|G| = d(G) + m|G|.$$  

**Proof.** By (3) and (Geroldinger & Ruzsa, 2009 [Lemma 2.1.2.]) we obtain

$$E(G) + (m - 1)|G| = D(G) + m|G| - 1 = d(G) + m|G|.$$  

Let $S = a_1 \cdot a_2 \cdot \ldots \cdot a_{D(G)-1}$ be a sequence of $D(G)-1$ non-zero elements in $G$.

Using the definition of $D(G)$, we may assume that $S$ does not contain any non-empty subsequence $S'$ such that $\sigma(S') = 0$. We put

$$T = a_1 \cdot a_2 \cdot \ldots \cdot a_{D(G)-1} \cdot 0 \cdot \ldots \cdot 0.$$  

We observe that the sequence $T$ does not contain $m$ disjoint non-empty subsequences $T_1, T_2, \ldots, T_m$ such that $\sigma(T_i) = 0$ and $|T_i| = |G|$ for $i \in [1, m]$. This implies that $E_{\Delta}(G) > D(G) + m|G| - 2$. Hence

$$E_{\Delta}(G) \geq E(G) + (m - 1)|G|.$$  

On the other hand, if $S$ is any sequence such that $|S| \geq E(G) + (m - 1)|G|$, then one can sequentially extract at least $m$ disjoint subsequences $S_1, \ldots, S_m$, such that $\sigma(S_i) = 0$ in $G$ and $|S_i| = |G|$. Thus

$$E_{\Delta}(G) \leq E(G) + (m - 1)|G|.$$
Corollary 4.2. For every finite Abelian group, the sequence \((E_m(G))_{m \geq 1}\) is an arithmetic progression with difference \(|G|\).

Corollary 4.3. If \(p\) is a prime and \(G=C_{p^{j_1}} \oplus \ldots \oplus C_{p^{j_k}}\) is a \(p\)-group, then for a natural \(m\) we have \(E_m(G) = mp\sum_{i=1}^{k} p^{j_i} - \sum_{i=1}^{k} (p^{j_i} - 1)\).

Proof. It follows from Remark 2.1 and Theorem 4.1.

We recall that by \(s_{(m)}(G)\) we denote the smallest \(t \in \mathbb{N} \cup \{\infty\}\) such that every sequence \(S\) over \(G\) of length \(t\) contains at least \(m\) disjoint non-empty subsequence \(S_1, S_2, \ldots, S_m\) such that \(\sigma(S_i) = 0, |S_i| = \exp(G)\).

Theorem 4.4. If \(G\) is a finite Abelian group, then
\[
\eta(G) + m \exp(G) - 1 \leq s_{(m)}(G) \leq s(G) + (m-1)\exp(G).
\]

Proof. The proof runs along the same lines as the proof of Theorem 4.1.

Let \(S = a_1, a_2, \ldots, a_{\eta(G)+1}\) be a sequence of \(\eta(G) - 1\) non-zero elements in \(G\). Using the definition of \(\eta(G)\), we may assume that \(S\) does not contain any non-empty subsequence \(S'\) such that \(\sigma(S') = 0, |S'| \leq \exp(G)\). We put

\[
T = a_1 \cdot a_2 \cdot \ldots \cdot a_{\eta(G)+1} \cdot 0 \cdot \ldots \cdot 0.
\]

We observe that sequence \(T\) does not contain \(m\) disjoint non-empty subsequence \(T_i, T_2, \ldots, T_m\) such that \(\sigma(T_i) = 0\) and \(|T_i| \leq \exp(G)\) for \(i \in [1, m]\). This implies that \(s_{(m)}(G) > \eta(G) + m \exp(G) - 1\). Hence \(s_{(m)}(G) = \eta(G) + m \exp(G) - 1\).

On the other hand, if \(S\) is any sequence over \(G\) such that \(|S| \geq s(G) + (m-1)\exp(G)\), then one can sequentially extract at least \(m\) disjoint subsequence \(S_1, \ldots, S_m\), such that \(\sigma(S_i) = 0\) in \(G\) and \(|S_i| = \exp(G)\). Thus, \(s_{(m)}(G) \leq s(G) + (m-1)\exp(G)\).

It was conjectured by Gao that for every finite Abelian group \(G\), one has \(\eta(G) + \exp(G) - 1 = s(G)\), see (Gao & Geroldinger, 2006 [Conjecture 6.5]). If this conjecture is true, then by Theorem 4.4 for every finite Abelian group \(G\) the equality \(s_{(m)}(G) = s(G) + (m-1)\exp(G)\) holds, i.e. the sequence \((s_{(m)}(G))_{m \geq 1}\) is an arithmetic progression with difference \(\exp(G)\). We will note that the equation \(\eta(G) + \exp(G) - 1 = s(G)\) is true for all finite Abelian groups of rank at most two, see (Girard & Schmid, 2019 [Theorem 2.3]). At this point it is worth mentioning that \(s(C_{n_1} \oplus C_{n_2}) = 2n_1 + 2n_2 - 3\) for all \(1 < n_1, n_2\), see (Girard & Schmid, 2019 [Theorem 2.3]).

Corollary 4.5. If \(G = C_{n_1} \oplus C_{n_2}\), where \(1 < n_1, n_2\), then for a natural \(m\) we have:
\[
E_m(G) = mn_1n_2 + n_1n_2 + 2n_1, \quad (5)
\]
\[
D_m(G) = mn_1n_2 + n_1n_2 - 1, \quad (6)
\]
\[
s_{(m)}(G) = (m+1)n_2 + 2n_1 - 3, \quad (7)
\]
\[
s_{(m)}(G) - D_m(G) = d(G). \quad (8)
\]

Proof. Equation (5) is a consequence of Remark 2.1 and Theorem 4.1; equation (6) follows from (Halter-Koch, 1992 [Proposition 5]). By applying (Girard & Schmid, 2019 [Theorem 2.3]) and Theorem 4.4, we can obtain (7). Equation (8) is a consequence of equations (6) and (7). Note that if \(G\) is an Abelian group, then \(D(G) - 1 = d(G)\) is the maximal length of a zero-sum free sequence over \(G\).
5. A generalization of the Kemnitz conjecture

Kemnitz’s conjecture states that every set \( S \) of \( 4n - 3 \) lattice points in the plane has a subset \( S' \) with \( n \) points whose centroid is also a lattice point. This conjecture was proven by Christian Reiher (2007). In order to prove the generalization of this theorem, we will use equation (7).

**Theorem 5.1.** Let \( n \) and \( m \) be natural numbers. Let \( S \) be a set of \( (m+3)n-3 \) lattice points in 2-dimensional Euclidean space. Then there are at least \( m \) pairwise disjoint sets \( S_1, S_2, \ldots, S_m \subseteq S \) with \( n \) points each, such that the centroid of each set \( S_i \) is also a lattice point.

**Proof.** As Harborth has already noted see (Edel et al., 2007; Harborth, 1973), \( s(C^2) \) is the smallest integer \( l \) such that every set of \( l \) lattice points in \( r \)-dimensional Euclidean space contains \( n \) elements which have a centroid in a lattice point. It is known that \( s(C^2) = 4n - 3 \) for all \( n \geq 1 \), see (Girard & Schmid, 2019, [Theorem 2.3]). By analogy, \( s_{\mathbb{N}}(C^2) \) is the smallest integer \( l \) such that every set \( S \) of \( l \) lattice points in \( r \)-dimensional Euclidean space has \( m \) pairwise disjoint subsets \( S_1, S_2, \ldots, S_m \) each of cardinality \( n \), the centroids of which are also lattice points. Finally, in the case of 2-dimensional Euclidean space, it is sufficient to use equation (7), from which we get \( s_{\mathbb{N}}(C^2) = (m+3)n-3 \). The last equality completes the proof of Theorem 5.1.

**Remark 5.2.** As we can see in Definition 3.1, Theorem 5.1 is also true when we replace sets with multisets.

6. Some results on \( s_{I,m}(G) \) constant

In this section, we will investigate zero-sum constants for finite Abelian groups. We start with \( s_{\mathbb{A}}(G), D_n(G), \eta(G) \) constants. Our main result of this section is Theorem 6.11. Olson (1969b) calculated \( s_{\mathbb{A}}(C^r_p) \) for a prime number \( p \). No precise result is known for \( s_{\mathbb{A}}(C^r_p) \), where \( n \geq 3 \). We need two technical lemmas:

**Lemma 6.1.** Let \( p \) be a prime number and \( n \geq 2 \). Then:

\[
s_{(n-1)p}(C^r_p) \leq (n+1)p-n.
\]

**Proof.** Let \( g \in C^r_p, i \in [1,(n+1)p-n] \). We embed group \( C^r_p \) into an Abelian group \( F \) which is isomorphic to \( C^{r-1} \). Let \( x \in F, \ x \not\in C^r_p \). Since \( D(C^{r-1}) = (n+1)p-n \) (see (Olson, 1969a) or Remark 2.1,(a)) there exists a zero-sum subsequence \( \prod_{i=1}^{(n-1)p-n}(x+g_i) \) of the sequence \( \prod_{i=1}^{(n-1)p-n}(x+g_i) \). But this is possible only if \( p \) divides \( |I| \). Rearranging subscripts, we may assume that \( g_1 + g_2 + \ldots + g_{\ell} = 0 \), where \( \ell \in [1,n] \). The thesis is achieved if \( \ell \in [1,n-1] \). If \( \ell = n \) we obtain a zero-sum sequence \( S = g_1 g_2 \ldots g_n \). Zero-sum sequence \( S \) contains a proper zero-sum subsequence \( S' \), since \( D(C^r_p) = np-(n-1) \), and thus a zero-sum subsequence of length not exceeding \( \frac{np}{2} \leq (n-1)p \).

**Corollary 6.2.** Let \( p \) be a prime. Then:

\[
s_{2p}(C^1_p) \leq 4p-3.
\]

**Lemma 6.3.** Let \( G \) be a finite Abelian group, \( k \in \mathbb{N}, k \geq \exp(G) \). If \( s_{[1,k]}(G) \leq s_{[1,k]}(G)+k \), then \( s_{[1,k],m}(G) \leq s_{[1,k],m}(G)+k \).
Proof. Let $S$ be a sequence over $G$ of length $s_k(G)+k$. The sequence $S$ contains a non-empty subsequence $S_i|S$ such that $\sigma(S_i)=0, [S_i] = [1,k]$, since $[S] \geq s_k(G) = s_k(G)$.

By the definition of $s_k(G)$ the remaining elements in $S$ contain $m$ disjoint non-empty subsequences $S_j|S$ such that $\sigma(S_j)=0, [S_j] = [1,k]$, where $i \in [1,m]$. Thus, we get $m+1$ non-empty disjoint subsequences $S_i|S$ such that $\sigma(S_i)=0, [S_i] = [1,k]$, where $i \in [0,m]$.

Corollary 6.4. Let $G$ be a finite Abelian group, $k \geq \exp(G)$. If $s_k(G) \leq s_k(G)+k$, then $s_k(G)+k \leq s_k(G)+nk$.

Proof. We use Lemma 6.3 and Remark 3.3.

Corollary 6.5. Let $G$ be a finite Abelian group, $k \geq \exp(G)$. Then:

\[
\begin{align*}
D_n(G) & \leq s_k(G) \leq s_k(G)+k(n-1), \\
D_n(G) & \leq s_k((G)\exp(G)) \leq \exp(G)(n-1).
\end{align*}
\]

Proof. We use Remark 3.3, Corollary 6.4 with $m=1$ and get (11). We put $k = \exp(G)$ in (11) and get (12).

Remark 6.6. It is known that:

1. $\eta(C_{\alpha \beta}^5) = 8\pi - 7$, if $n = 3^\alpha 5^\beta$, with $\alpha, \beta \geq 0$,
2. $\eta(C_{\alpha \beta}^7) = 7\pi - 6$, if $n = 3^\alpha 2^\beta$, with $\alpha \geq 1$,
3. $\eta(C_{\alpha \beta}^8) = 8, \eta(C_{\alpha \beta}^9) = 17, \eta(C_{\alpha \beta}^{10}) = 39, \eta(C_{\alpha \beta}^{11}) = 89, \eta(C_{\alpha \beta}^{12}) = 223$ (Girard, 2018).

Corollary 6.7. We have:

1. $D_n(C_{\alpha \beta}^3) \leq nm+7n-7$, if $n = 3^\alpha 5^\beta$, with $\alpha, \beta \geq 0$,
2. $D_n(C_{\alpha \beta}^5) \leq nm+6n-6$, if $n = 3^\alpha 2^\beta$, with $\alpha \geq 1$,
3. $D_n(C_{\alpha \beta}^7) \leq 3m+36, D_n(C_{\alpha \beta}^8) \leq 3m+86, D_n(C_{\alpha \beta}^9) \leq 3m+220$.

Proof. By Corollary 6.5 and Remark 6.6.

In the next Lemma, we collect several useful properties on the Davenport constant.

Lemma 6.8. Let $G$ be a non-trivial finite Abelian group and $H$ be a subgroup of $G$. Then:

\[
D(H)+D(G/H)-1 \leq D(G) \leq D(H)+D(G/H) \leq D(H)D(G/H). \tag{13}
\]

Proof. The inequality $D(H)+D(G/H)-1 \leq D(G)$ is proven in (Halter-Koch, 1992 [Proposition 3 (i)]). Now we prove the inequality $D(G) \leq D(H)+D(G/H)$ on the same lines as in (Delorme, Ordaz & Quiroz, 2001), we include the proof for the sake of completeness.

If $|S| \geq D(H)+D(G/H)$ is any sequence over $G$, then one can, by definition, extract at least $D(H)$ disjoint non-empty subsequences $S_1, ..., S_{D(H)}|S$ such that $\sigma(S_i) \in H$. Since $T = \prod_{i=1}^{D(H)} \sigma(S_i)$ is a sequence over $H$ of length $D(H)$, there thus exists a non-empty subset $I \subseteq [1,D(H)]$ such that $T' = \prod_{i \in I} \sigma(S_i)$ is a zero-sum subsequence of $T$.

We obtain that if $S' = \prod_{i \in I} S_i$ is a non-empty zero-sum subsequence of $S$. The inequality $D(H)+D(G/H) \leq D(H)D(G/H)$ follows from Remark 3.3.
Theorem 6.9. For an Abelian group $C_p \oplus C_p \oplus C_p$ such that $p | n_j n_i \in \mathbb{N}$, where $p$ is a prime number, we have
\[ n_i + n_j + p - 2 \leq D(C_p \oplus C_p \oplus C_p) \leq D_{\frac{n_i}{p}, \frac{n_j}{p}}(C_p^k) \leq 2n_i + 2n_j - 3. \] (14)

Proof. If $G = C_p \oplus C_p \oplus C_p$, such that $p | n_i n_j \in \mathbb{N}$, then $\exp(G) = n_i$. Note that $n_i + n_j + p - 2 = D^*(G) < D(G)$. Denoting by $H$ a subgroup of $G$ such that $H = C_2^p \oplus C_2^p$. The quotient group $G/H \cong C_p \oplus C_p \oplus C_p$. By Lemma 6.8 we get
\[ D(G) \leq D_{\frac{n_i}{p}, \frac{n_j}{p}}(C_p^k) \approx (15) \]

By (11) (with $m = \frac{n_i}{p} \geq 1$, $k = 2p$) and Corollary 6.2, we get
\[ D(G) \leq S_{\frac{n_i}{p}}(C_p^k) + 2p \left( \frac{n_i}{p} + \frac{n_j}{p} - 2 \right) \leq 2p \left( \frac{n_i}{p} + \frac{n_j}{p} - 2 \right) + 4p - 3 = 2n_i + 2n_j - 3. \] (16)

Our next goal is to generalize (Delorme, Ordaz & Quiroz, 2001 [Theorem 3.2]) to the case $r(\geq 3)$.

Theorem 6.10. Let $H$, $K$ and $L$ be Abelian groups of orders $|H| = h$, $|K| = k$ and $|L| = l$. If $G = H \oplus K \oplus L$ with $h | k | l$. Let $\Omega(h)$ denote the total number of prime factors of $h$. Then:
\[ S_{\frac{n_i}{p}}(G) \leq 2^{\Omega(h)}(2l + k + h) - 3. \] (17)

Proof. The proof will be inductive. If $h = 1$, then by (Delorme, Ordaz & Quiroz, 2001 [Theorem 3.2]) we have
\[ S_{\frac{n_i}{p}}(G) = S_{\frac{n_j}{p}}(K \oplus L) \leq 2l + k - 2 = 2^{\Omega(1)}(2l + k + 1) - 3. \] (18)

Assume that $h > 1$ and let $p$ be a prime divisor of $h.$ Let $H_1$ be a subgroup of $H$, $K_1$ be a subgroup of $K$, $L_1$ be a subgroup of $L$, with indices $|H:H_1| = |K:K_1| = |L:L_1| = p$. Put $h = ph_1$, $k = pk_1$, $l = pl_1$ and $Q = H_1 \oplus K_1 \oplus L_1$. Assume inductively that theorem is true for $Q$ i.e.
\[ S_{\frac{n_i}{p}}(Q) \leq 2^{\Omega(h_1)}(2l_1 + k_1 + h_1) - 3. \] (19)

Let $s = 2^{\Omega(h)}(2l + k + h) - 3$ and $S = g_1, g_2, \ldots, g_s$ be a sequence of $G$. We shall prove that there exists a subsequence of $S$ with a length smaller than or equal to $2^{\Omega(h)}$ and a zero sum. Let $b_i = g_i + Q \in G/Q$. We consider the sequence $\prod_{i=1}^{s} b_i$ of length $s$. The quotient group $G/Q$ is isomorphic to $C_p^s$ and
\[ s = 2p(2^{\Omega(h)})(2l_1 + k_1 + h_1) - 2 + 4p - 3. \] (20)

Therefore, by Corollary 6.2 there exists at least $j_0$ pairwise disjoint sets $I_j \subseteq [1, s], |I_j| \leq 2p$, where
\[ j \leq j_0 = 2^{\Omega(h)}(2l_1 + k_1 + h_1) - 1, \] (21)

such that each sequence $\prod_{i \in I} b_i$ has a zero sum in $G/Q$. In other words
\[ \alpha\left(\prod_{i \in I} g_i\right) = \sum_{i \in I} g_i \in Q. \] By induction assumption for $Q$ there exists $J \subseteq [1, j_0]$ with $|J| \leq 2^{\Omega(h)}$ such that $\sum_{i \in J} \alpha\left(\prod_{i \in J} g_i\right) = 0$. Thus, we obtain a zero-sum
subsequence of \( S \) in \( G \) of length not exceeding \( \sum_{j=1}^{s} |I_j| \leq 2^{\alpha(h_1)} \cdot 2^p = 2^{\alpha(h_1)} \), which ends the inductive proof.

**Theorem 6.11.** Let \( H_1, H_2, \ldots, H_s \) be Abelian groups of orders \( |H_i| = h_i \). If \( n \geq 2 \) and \( G = H_1 \oplus H_2 \oplus \ldots \oplus H_s \) with \( h_i \), then

\[
s_{(n-1)p_c/h_c}(G) \leq (n-1)^{\alpha(h_1)}(2(h_{n-1}) + (h_{n-1} - 1) + \ldots + (h_{1-1}) + 1).
\]

(22)

**Proof.** We proceed by induction on \( n \) and \( h_i \).

If \( n = 2 \), then the inequality (22) holds by (Delorme, Ordaz & Quiroz, 2001 [Theorem 3.2]). Namely

\[
s_{(2,1)p_c/h_c}(H_1 \oplus H_2) = s_{h_c}(H_1 \oplus H_2) \leq 2h_2 + 2h_1 - 2 = (2-1)^{\alpha(h_1)}(2(h_2 - 1) + (h_1 - 1) + 1).
\]

(23)

Suppose that the inequality (22) holds for fixed \( n-1 \geq 2 \):

\[
s_{(n-2)p_c/h_c}(H_1 \oplus \ldots \oplus H_{n-1}) \leq (n-2)^{\alpha(h_1)}(2(h_{n-1} - 1) + (h_{n-1} - 2) + \ldots + (h_1 - 1) + 1).
\]

(24)

If \( n \geq 3 \) and \( h_1 = 1 \), then \( G \) and \( H_2 \oplus \ldots \oplus H_s \) are isomorphic. By (2):

\[
s_{(n-1)p_c/h_c}(G) = s_{(n-1)p_c/h_c}(H_1 \oplus \ldots \oplus H_s) \leq s_{(n-2)p_c/h_c}(H_1 \oplus \ldots \oplus H_s).
\]

(25)

Thus by the induction hypothesis (24):

\[
s_{(n-1)p_c/h_c}(G) \leq (n-2)^{\alpha(h_1)}(2(h_{n-1} - 1) + (h_{n-1} - 2) + \ldots + (h_1 - 1) + 1) \leq (n-1)^{\alpha(h_1)}(2(h_{n-1} - 1) + (h_{n-1} - 1) + \ldots + (h_1 - 1) + (1-1) + 1).
\]

Therefore (22) holds.

Suppose that the inequality (22) holds for fixed \( n \geq 3 \) and fixed \( h_i \), such that \( h_i > h_1 \geq 1 \):

\[
s_{(n-1)p_c/h_c}(G) \leq (n-1)^{\alpha(h_1)}(2(h_{n-1} - 1) + (h_{n-1} - 1) + \ldots + (h_1 - 1) + 1).
\]

(27)

Let \( p \) be a prime divisor of \( h_1 \). Let \( H_1' \) be a subgroup of index \( p \) of a group \( H_i \). Put \( h_i = ph_i' \) and \( Q = H_1' \oplus H_2' \oplus \ldots \oplus H_s' \). By inductive assumption, the inequality (22) holds for \( Q \):

\[
s_{(n-1)p_c/h_c}(Q) \leq 2^{\alpha(h_1)}(2(h_{n-1} - 1) + (h_{n-1} - 1) + \ldots + (h_1 - 1) + 1).
\]

(28)

We put \( s = (n-1)^{\alpha(h_1)}(2(h_{n-1} - 1) + (h_{n-1} - 1) + \ldots + (h_1 - 1) + 1) \) and let \( S = g_1, g_2, \ldots, g_s \) be a sequence of \( G \).

We shall prove that there exists a subsequence of \( S \) with a length smaller than or equal to \( (n-1)^{\alpha(h_1)}h_1 \) and a zero sum. Let \( b_i = g_i + Q, 1 \leq i \leq s \), be the sequence of \( G/Q \).

The quotient group \( G/Q \) is isomorphic to \( C_{h_1}^* \) and

\[
s = (n-1)^{\alpha(h_1)}(2(h_{n-1} - 1) + (h_{n-1} - 1) + \ldots + (h_1 - 1) + 1) \geq \frac{p(n-1)^{\alpha(h_1)}(2(h_{n-1} - 1) + (h_{n-1} - 1) + \ldots + (h_1 - 1) + 1) + (n-1)^{\alpha(h_1)}(2(h_{n-1} - 1) + (h_{n-1} - 1) + \ldots + (h_1 - 1) + 1) + 2p - n}{n}
\]

(29)

Therefore, by Lemma 6.1 there exists at least \( j_0 \) pairwise disjoint sets \( I_j \subseteq [1, s] \) with \( |I_j| \leq (n-1)p \) and

\[
(j \leq j_0 = (n-1)^{\alpha(h_1)}(2(h_{n-1} - 1) + \ldots + (h_1 - 1) + 1),
\]

(30)

such that sequence \( \prod_{p \in I_j} b_i \) has a zero sum in \( G/Q \).
In another words $\sigma(\prod_{i=1}^{n} g_i) = \sum_{i=1}^{n} g_i \in Q$. By induction assumption (22) for $Q$ there exists $J \subseteq [1,j_0]$ with $|J| \leq (n-1)\sigma^k(h_n^*)$ such that $\sum_{j \in J} \sigma(\prod_{i=1}^{n} g_i) = 0$.

Thus, we obtain a zero sum subsequence of $S$ in $G$ of length not exceeding

$$\sum_{j \in J} |J| \leq (n-1)\sigma^k(h_n^*)(n-1)p = (n-1)\sigma^k(h_n^*).$$

7. The smooth numbers

First, we recall the notation of a smooth number. Let $F = \{q_1, q_2, \ldots, q_r\}$ be a subset of positive integers. A positive integer $k$ is said to be smooth over a set $F$ if $k = q_1 \cdot q_2 \cdot \ldots \cdot q_r$, where $e_i$ are non-negative integers.

Remark 7.1. Let $n \in \mathbb{N}$. Each smooth number over a set $\{q_1^e, q_2^e, \ldots, q_r^e\}$ is an $n$-th power of a suitable smooth number over the set $\{q_1, q_2, \ldots, q_r\}$.

Definition 7.2. Let $\{p_1, p_2, \ldots, p_s\}$ be a set of distinct prime numbers. By $c(n_1, n_2, \ldots, n_s)$, we denote the smallest $t \in \mathbb{N}$ such that every sequence $M$ of smooth numbers over a set $\{p_1, p_2, \ldots, p_s\}$, of length $t$ has a non-empty subsequence $N$ such that the product of all the terms of $N$ is a smooth number over a set $\{p_1^e, p_2^e, \ldots, p_s^e\}$.

In the next theorem, we use notation $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_s}$ instead of notation $C_{n_1} \oplus C_{n_2} \oplus \ldots \oplus C_{n_s}$, these two structures are isomorphic to one another.

Theorem 7.3. Let $n_1, n_2, \ldots, n_s$ be integers such that $1 < n_1 | n_2 | \ldots | n_s$. Then:

$$c(n_1, n_2, \ldots, n_s) = D(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_s}).$$

Proof. It follows on the same lines as the proof of (Chintamani et al., 2012 [Theorem 1.6]). First, we prove that

$$c(n_1, n_2, \ldots, n_s) \leq D(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_s}).$$

We put $l = D(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_s})$. Let $M = (m_1, m_2, \ldots, m_l)$ be a sequence of smooth numbers with respect to $F = \{p_1, p_2, \ldots, p_s\}$.

For all $i \in [1, l]$, we have $m_i = p_1^{e_1(i)} \cdot p_2^{e_2(i)} \cdot \ldots \cdot p_s^{e_s(i)}$, where $e_j(i)$ are non-negative integers.

We associate each $m_i$ with $a_i \in \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_s}$ under the homomorphism:

$$\Phi : \{p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_s^{e_s} : e_j \geq 0, e_j \in \mathbb{Z}\} \rightarrow \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_s},$$

$$\Phi(p_1^{e_1} \cdot p_2^{e_2} \cdot \ldots \cdot p_s^{e_s}) = ([e_1], [e_2], \ldots, [e_s]).$$

Hence

$$\Phi(m_i) = ([e_{1,i}], [e_{2,i}], \ldots, [e_{s,i}]).$$

Thus, we get a sequence $S = a_1, a_2, \ldots, a_l$ of elements of the group $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_s}$ of length $l = D(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_s})$. Therefore, there exists a non-empty zero sum subsequence $T$ of $S$ in $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_s}$. Let $T = a_1^*, a_2^*, \ldots, a_k^*$, where

$$\sum_{i=1}^{k} e_{i,k} \equiv 0 (\text{mod } n_k),$$

where $k \in [1, r]$. 

https://doi.org/10.37705/TechTrans/e2020027
Consider the subsequence $N$ of $M$ corresponding to $T$. We have $N = (m_{i'}, m_{i''}, \ldots, m_i)$ and by equation (35), we get

$$
\prod_{i'=1}^{l} m_{i'} = \prod_{i'=1}^{l} p_i^{\sum_{j=1}^{l} gj} = \prod_{i'=1}^{l} (p_i^{n_i}),
$$

for some integers $l_i \geq 0$. Thus, the product $\prod_{i'=1}^{l} m_{i'}$ of all the terms of $N$ is a smooth number over a set $\{p_1^{n_1}, p_2^{n_2}, \ldots, p_i^{n_i}\}$. By definition of $c(n_i, n_j, \ldots, n_t)$, we get inequality (32).

On the other hand, we will now prove that

$$
c(n_i, n_j, \ldots, n_t) \geq D(\mathbb{Z}_{n_i} \oplus \mathbb{Z}_{n_j} \oplus \cdots \oplus \mathbb{Z}_{n_t}).
$$

Let $l = c(n_i, n_j, \ldots, n_t)$ and $S = a_1 a_2 \cdots a_t$ be a sequence of elements of $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ of length $l$, where $a_i = ([e_{i1}], [e_{i2}], \ldots, [e_{i \ell}])$.

We put $m_i = p_1^{e_{i1}} p_2^{e_{i2}} \cdots p_t^{e_{i \ell}}$. The sequence $M = (m_1, m_2, \ldots, m_t)$ of integers is a sequence of smooth numbers over a set $F$.

By definition of $l = c(n_i, n_j, \ldots, n_t)$, there exists a non-empty subsequence $N = (m_{i'}, m_{i''}, \ldots, m_i)$ of $M$ such that

$$
\prod_{i'=1}^{l} m_{i'} = \prod_{i'=1}^{l} (p_i^{n_i}),
$$

for some integers $l_i \geq 0$. The subsequence $T$ of $S$ corresponding to $N$ will sum up to the identity in $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_t}$. Therefore, (37) holds and we obtain (31).

8. Future work and the non-Abelian case

The constant $E(G)$ has received a lot of attention over the last ten years. For example, one direction towards weighted zero-sum problems, see (Grynkiewich, 2013 [Chapter 16]). The second direction towards non-Abelian groups.

Let $G$ be any additive finite group. Let $S = (a_1, a_2, \ldots, a_n)$ be a sequence over $G$. We say that the sequence $S$ is a zero-sum sequence if there exists a permutation $\sigma: [1, n] \rightarrow [1, n]$ such that $0 = a_{\sigma(1)} + \cdots + a_{\sigma(n)}$. For a subset $I \subseteq \mathbb{N}$, let $s_i(G)$ denote the smallest $i \in \mathbb{N} \cup \{0, \infty\}$ such that every sequence $S$ over $G$ of length $|S| \geq i$ has a zero-sum subsequence $S'$ of length $|S'| \leq i$. The constants $D(G) := s_0(G)$ and $E(G) := s_{|G|}(G)$ are classical invariants in zero-sum theory (independently of whether $G$ is Abelian or not). We recall that for a given finite group $G$, we denote by $d(G)$, the maximal length of a zero-sum free sequence over $G$. We call $d(G)$ the small Davenport constant.

Remark 8.1. In the Abelian case $d(G) = D(G) - 1$. Note that in the case of non-Abelian groups, $d(G)$ can be strictly smaller than $D(G) - 1$, see (Geroldinger & Ruzsa, 2009 [Chapter 2]).

Theorem 8.2. If $G$ is a finite group of order $|G|$, then

$$
d(G) + m|G| \leq E_m(G) \leq E(G) + (m-1)|G|.
$$

Proof. Let $S = (a_1, a_2, \ldots, a_{d(G)})$ be a sequence of $d(G)$ non-zero elements in $G$. Using the definition of $d(G)$, we may assume that $S$ does not contain any non-empty subsequence $S'$ such that $\sigma(S') = 0$. We put

$$
T = (a_1, a_2, \ldots, a_{d(G)}, 0, \ldots, 0)
$$

where $v_0(T) = m|G| - 1$. 

https://doi.org/10.37705/TechTrans/e2020027
We observe that the sequence $T$ does not contain $m$ disjoint non-empty subsequences $T_1, T_2, \ldots, T_m$ of $T$ such that $\sigma(T_i) = 0$ and $|T_i| = |G|$ for $i \in [1, m]$. This implies that $E_m(G) > d(G) + m|G| - 1$. Hence

$$E_m(G) \geq d(G) + m|G|. \tag{41}$$

On the other hand, if $S$ is any sequence over $G$ such that

$$|S| \geq E(G) + (m-1)|G|,$$

then one can sequentially extract at least $m$ disjoint subsequences $S_1, \ldots, S_m$ of $S$, such that $\sigma(S_i) = 0$ in $G$ and $|S_i| = |G|$. Thus

$$E_m(G) \leq E(G) + (m-1)|G|. \tag{42}$$

\textbf{Corollary 8.3.} If $m \to \infty$, then $E_m(G) = m|G|$.

\textit{Proof.} If $m \to \infty$, then $\frac{E_m(G)}{m} \to |G|$ by the inequality (39).

We now give an application of Theorem 8.2. The formula $E(G) = d(G) + |G|$ was proved for all finite Abelian groups and for some classes of finite non-Abelian groups (see equation (3) and (Bass, 2007; Han, 2015; Han & Zhang, 2019; Oh & Zhong, 2019)). Thus

$$E_m(G) = d(G) + m|G|$$

holds for finite groups in the following classes: Abelian groups, nilpotent groups, groups in the form $C_n \rtimes C_m$, where $m, n \in \mathbb{N}$, dihedral and dicyclic groups and all non-Abelian groups of order $pq$ with $p$ and $q$ prime. Therefore, the following conjecture can be proposed:

\textbf{Corollary 8.4.} (See (Bass, 2007 [Conjecture 2]))

For any finite group $G$, we have the equation $E_m(G) = d(G) + m|G|$.  

9. Conclusion

This paper makes a contribution to the theory of additive combinatorics. It provides an overview of the state of knowledge of the zero-sum problems and can be considered as an introduction to this theory. We have proven that if $G$ is an Abelian group, then $E_m(G) = d(G) + m|G|$. We have studied the asymptotic behaviour of the sequences $(E_m(G))_{m \geq 1}$ and $(\eta_m(G))_{m \geq 1}$. For a prime $p$ and a natural $n \geq 2$, we have derived the inequality $s_{(n+1)}(C_p^\infty) \leq (n+1)p-n$. We have proven a generalization of Kemnitz’s conjecture. We have applied the Davenport constant to smooth numbers. Finally, we have shown some results in the non-Abelian case.

\textbf{2010 Mathematics Subject Classification:} Primary 11P70; Secondary 11B50
References


