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Squared basis operators related to Bessel functions

Monika Herzog[1]

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Abstract
Recent studies on linear positive operators have led to the investigation of approximation properties of Szász–Mirkyan operators related to the modified Bessel function of order 0. In this paper, we analyse the asymptotic behavior of these operators, convergence theorems, Voronovskaya and Grüss-Voronovskaya type results. A comparative assessment with classical Szász–Mirakyan operators is also presented. These results may have an impact on wide branches of knowledge, such as probability theory, statistics, physical chemistry, optics, and computer science, especially signal processing.

Keywords Squared basis · Moment generating function · Bessel function · Voronovskaya-type results

Mathematics Subject Classification 41A25 · 41A30 · 41A36

1 Introduction

In 1941, G. M. Mirakyan [19] proposed new approximating operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right)$$

for function $f : \mathbb{R}_0 \to \mathbb{R}$, such that the above series is absolutely convergent. It was a generalization of approximation $f \in C[0,1]$ by basic Bernstein operators. Later, independently, J. Favard [7] and O. Szász [22] published similar results. Now, we often call these operators Szász–Mirakyan or sometimes Szász–Mirakyan–Favard operators.

In view of paper [16], it occurs that both Szász–Mirakyan and Bernstein operators belong to a wide class, so-called exponential operators. That means, they can be

✉ Monika Herzog
mherzog@pk.edu.pl

1 Department of Applied Mathematics, Cracow University of Technology, Warszawska 24, 31155 Cracow, Poland

interpreted as integral operators with kernel $W(n, t, u)$,

$$L_n(f; x) = \int_a^b W(n, x, u) f(u) du,$$

which fulfill the normalization condition

$$\int_a^b W(n, x, u) du = 0,$$

and satisfy the following differential equation

$$(L_n(f; x))' = \frac{n}{p(x)} L_n(f(e_1 - xe_0); x),$$

where p is a positive analytic function defined on (a, b) and $e_i(t) = t^i$ for $i \in \mathbb{N}_0$.

In 2003, King's modification of the original Bernstein operators [18] initiated a new approach to the examination of properties of these operators. The modification assumes that the new operators preserve test functions e_0 and e_2. Based on King's idea, new operators with a smaller error of approximation appeared, for example [2], [5], [6], [15].

In this paper, we will consider a special case of operators introduced in paper [12], that means

$$A_n^0(f; x) = \frac{1}{I_0(nx)} \sum_{k=0}^{\infty} \frac{\left(\frac{nx}{2}\right)^{2k}}{(\Gamma(k+1))^2} f\left(\frac{2k}{n}\right), \tag{1}$$

for $n \in \mathbb{N}$, $f \in C_*(\mathbb{R}_0)$ and the following modified Bessel function of the first kind

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2k}}{(\Gamma(k+1))^2}.$$

We focus on these operators because their rescaling version comes in the King-type operator related to squared Szász–Mirakyan basis (see [15]). Moreover, in Lemma 2.3 [15] the error of approximation by these operators was proved to be smaller than by the classical Szász–Mirakyan ones, for uniformly continuous functions. Motivated by the favorable approximation behavior of A_n^0, we will examine the error of convergence for these operators in $C_*(\mathbb{R}_0)$. Also, we will consider the Voronovskaya-type results for our operators. Finally, we will investigate differences between A_n^0 and the basic Szász–Mirakyan operators. We shall present a new approach based on the newest results in the area of research.

2 Auxiliary results

Let us denote the space of all real-valued continuous functions on $\mathbb{R}_0 = [0, \infty)$ by $C(\mathbb{R}_0)$, and the Banach space of all continuous and bounded functions on $\mathbb{R}_0$ endowed with the sup-norm $\|\cdot\|_\infty$ by $C_b(\mathbb{R}_0)$. Moreover, we shall also consider the following closed subspace of $C_b(\mathbb{R}_0)$

$$C_*(\mathbb{R}_0) = \{f \in C_b(\mathbb{R}_0) \mid \lim_{x\to\infty} f(x) \in \mathbb{R}\},$$

and for $k \in \mathbb{N}$ we shall denote

$$C_*^{(k)}(\mathbb{R}_0) = \{f \in C_*(\mathbb{R}_0) : f', f'', ..., f^{(k)} \in C_*(\mathbb{R}_0)\}.$$

Remark 1 As usual, we denote $e_n(t) = t^n$ the monomial of degree $n \in \mathbb{N}$. Moreover, we use the software Mathematica for some calculations.

Whenever we investigate approximation operators, we are interested in the convergence of these operators to a given function f. One of possible answers to the question presents Theorem 2.1 from paper [4].

Theorem 1 *[[4]] Let us consider positive linear operators (L_n), L for $n \in \mathbb{N}$. Suppose that for $s \in \mathbb{R}$ and $x \in \mathbb{R}_0$*

$$\lim_{n\to\infty} L_n(e^{ist}; x) = L(e^{ist}; x).$$

Then

$$\lim_{n\to\infty} L_n(f(t); x) = L(f(t); x)$$

for all $f \in C_b(\mathbb{R}_0)$ and $x \in \mathbb{R}_0$.

We start by setting the moment generating function (MGF) for our operators, which is a consequence of the Lemma 5 from paper [13].

Proposition 2 *If $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, then the moment generating function of A_n^0 is given by*

$$A_n^0(\exp_A; x) = \frac{I_0(nxe^{\frac{A}{n}})}{I_0(nx)}$$

for A in some open neighborhood of 0.

For convenience of the reader, we also recall Lemmas 1 and 2 according to paper [12] and calculate some additional moments of our operators.

Lemma 1 *For each $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, we have*

$$
\begin{aligned}
A_n^0(e_0; x) &= 1, \\
A_n^0(e_1; x) &= x \frac{I_1(nx)}{I_0(nx)}, \\
A_n^0(e_2; x) &= x^2, \\
A_n^0(e_3; x) &= x^3 \frac{I_1(nx)}{I_0(nx)} + \frac{2x^2}{n}, \\
A_n^0(e_4; x) &= x^4 + \frac{4x^3}{n} \frac{I_1(nx)}{I_0(nx)} + \frac{4x^2}{n^2}.
\end{aligned}
$$

Moreover, if we denote the central moments of operators A_n^0 by

$$\mu_{n,r}(x) = A_n^0((e_1 - xe_0)^r; x),$$

then we have the following relations

Lemma 2 *For each $n \in \mathbb{N}$ and $x \in \mathbb{R}_0$, we get*

$$
\begin{aligned}
\mu_{n,0}(x) &= e_0, \\
\mu_{n,1}(x) &= x \left(\frac{I_1(nx)}{I_0(nx)} - 1 \right), \\
\mu_{n,2}(x) &= 2x^2 \left(1 - \frac{I_1(nx)}{I_0(nx)} \right), \\
\mu_{n,3}(x) &= 4x^3 \left(\frac{I_1(nx)}{I_0(nx)} - 1 \right) + \frac{2x^2}{n}, \\
\mu_{n,4}(x) &= 8x^4 \left(1 - \frac{I_1(nx)}{I_0(nx)} \right) + \frac{4x^3}{n} \left(\frac{I_1(nx)}{I_0(nx)} - 2 \right) + \frac{4x^2}{n^2}.
\end{aligned}
$$

First, we apply Theorem 1 to achieve the convergence of operators A_n^0.

Theorem 3 *Let $f \in C_b(\mathbb{R}_0)$, $x \in \mathbb{R}_0$ and m be a non-negative integer. Then, the following convergence holds true*

$$\lim_{n \to \infty} A_n^0(f(t); x) = f(x), \tag{2}$$

$$\lim_{n \to \infty} A_n^0 \left(f \left(\frac{t}{n} \right); nx \right) = f(x), \tag{3}$$

$$\lim_{n \to \infty} A_{mn}^0 \left(f(nt); \frac{x}{n} \right) = A_m^0(f(t); x). \tag{4}$$

Proof By the definition of operators A_n^0 and the MGF for them, we have

$$A_n^0(e^{ist}; x) = \frac{I_0(nxe^{\frac{is}{n}})}{I_0(nx)}.$$

Using the following property of the modified Bessel function

$$\lim_{z\to+\infty} \frac{I_\nu(z)\sqrt{2\pi z}}{e^z} = 1, \tag{5}$$

we get

$$\lim_{n\to\infty} A_n^0(e^{ist}; x) = \lim_{n\to\infty} e^{nx(e^{\frac{is}{n}}-1)}e^{-\frac{is}{2n}} = e^{isx} = Id(e^{ist}; x),$$

where Id is the identity operator. Hence, by Theorem 1, we conclude the convergence of our operators A_n^0 to the approximated function f.
Now, we proceed to prove the next convergence (3). Let us consider $A_n^0(e^{\frac{ist}{n}}; nx)$. Applying Proposition 2, we have

$$A_n^0(e^{\frac{ist}{n}}; nx) = \frac{I_0(n^2xe^{\frac{is}{n^2}})}{I_0(n^2x)}.$$

Taking into account (5), we get

$$\lim_{n\to\infty} A_n^0(e^{\frac{ist}{n}}; nx) = \lim_{n\to\infty} e^{n^2x(e^{\frac{is}{n^2}}-1)}e^{-\frac{is}{2n^2}} = e^{isx} = Id(e^{ist}; x).$$

In virtue of Theorem 1, we get limit (3) in the theses of our theorem.
If we consider limit (4), we have to write the following expression

$$A_{mn}^0\left(e^{isnt}; \frac{x}{n}\right) = \frac{I_0(mxe^{\frac{is}{m}})}{I_0(mx)}$$

and using (5), we achieve

$$\lim_{n\to\infty} A_{mn}^0\left(e^{isnt}; \frac{x}{n}\right) = A_m^0(e^{ist}; x).$$

Theorem 1 leads to convergence (4). □

Remark 2 It is easy to observe that the sequence of operators A_n^0 has properties (C_1) and (C_2) defined in paper [3], with $R_m = A_m^0$ and $Q_m = Id$, respectively.

According to Example 2 ([3]), it occurs that operators A_n^0 and S_n have similar properties. Although, our operators are not an exponential-type.

3 Estimation theorems and Voronovskaya-type results

Now, we move onto the estimation of the error of approximation using Theorem 2.1 from paper [14]. Similar results we can find for example in papers [2], [9], [11], [17]. Note that in the proof of Theorem 5, among other properties, we utilize an interesting multiplication connection for modified Bessel functions.

Theorem 4 *[[14]] Let* $L_n : C_*(\mathbb{R}_0) \to C_*(\mathbb{R}_0)$ *be a sequence of positive linear operators with*

$$a_{m,n} := \|L_n(\exp_{-m}) - \exp_{-m}\|_\infty, \quad m \in \{0, 1, 2\}$$

where

$$\lim_{n\to\infty} a_{m,n} = 0, \quad m \in \{0, 1, 2\}.$$

Then, for $f \in C_*(\mathbb{R}_0)$*, we have*

$$\|L_n(f) - f\|_\infty \le a_{0,n}\|f\|_\infty + (2 + a_{0,n})\,\omega^*\left(f, \sqrt{a_{0,n} + 2a_{1,n} + a_{2,n}}\right),$$

where

$$\omega^*(f, \delta) := \sup\{|f(x) - f(t)| : x, t \ge 0, |e^{-x} - e^{-t}| \le \delta\}.$$

Remark 3 In paper [2] was noticed that we do not need assumptions about the convergence of $a_{n,m}$ to 0.

For our operators, we have the following estimation

Theorem 5 *Let* $f \in C_*(\mathbb{R}_0)$*. Then the following estimation holds true*

$$\|A_n^0(f) - f\|_\infty \le 2\,\omega^*\left(f, \sqrt{2a_{1,n} + a_{2,n}}\right),$$

where

$$a_{i,n} = \|A_n^0(\exp_{-i}) - \exp_{-i}\|_\infty$$

and

$$\lim_{n\to\infty} a_{i,n} = 0$$

for $i \in \{1, 2\}$.

Proof By Lemma 1, we notice that operators A_n^0 preserve constants; therefore, in Theorem 4 coefficient $a_{0,n} = 0$. Next, we exam the limit of $a_{1,n}$ and $a_{2,n}$ as $n \to \infty$. The MGF of the operators yields

$$A_n^0(\exp_{-1}; x) = \frac{I_0(nxe^{-\frac{1}{n}})}{I_0(nx)}, \quad A_n^0(\exp_{-2}; x) = \frac{I_0(nxe^{-\frac{2}{n}})}{I_0(nx)}.$$

By Theorem 3, we immediately get for $x \in \mathbb{R}_0$

$$\lim_{n\to\infty} A_n^0(\exp_{-1}; x) = e^{-x}, \quad \lim_{n\to\infty} A_n^0(\exp_{-2}; x) = e^{-2x}.$$

In order to estimate the uniform norm $a_{i,n}$, we recall a multiplication property for modified Bessel functions of the first kind. For $\theta, \nu \in \mathbb{C}$ such that $|\theta^2 - 1| < 1$, we have

$$\theta^{-\nu} I_\nu(\theta z) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{(\theta^2 - 1)z}{2} \right)^k I_{\nu+k}(z).$$

Moving onto estimation of $a_{1,n}$, we apply the above relation for $\theta = e^{-\frac{1}{n}}$, $z = nx$ and $\nu = 0$

$$\begin{aligned} A_n^0(\exp_{-1}; x) - e^{-x} &= \frac{I_0(nxe^{-\frac{1}{n}})}{I_0(nx)} - e^{-x} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{(e^{-\frac{2}{n}} - 1)nx}{2} \right)^k \frac{I_k(nx)}{I_0(nx)} - e^{-x}. \end{aligned}$$

Reducing the first items of the series, we obtain

$$A_n^0(\exp_{-1}; x) - e^{-x} = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{(e^{-\frac{2}{n}} - 1)nx}{2} \right)^k \frac{I_k(nx)}{I_0(nx)} - (e^{-x} - 1).$$

Due to the monotonicity of I_ν with respect to ν, we can write $\frac{I_k(nx)}{I_0(nx)} < 1$ for $k \geq 1$. Now, we get

$$A_n^0(\exp_{-1}; x) - e^{-x} \leq \exp\left(\frac{-x(1 - e^{-\frac{2}{n}})}{\frac{2}{n}} \right) - e^{-x}.$$

Using Lemma 3.1 from paper [14] with $\alpha_n = \frac{1-e^{-x_n}}{x_n}$ and $x_n = \frac{2}{n}$, we have the following estimation

$$A_n^0(\exp_{-1}; x) - e^{-x} \leq \frac{1}{ne}$$

On the other hand, for the upper bound of the expression, we notice that $A_n^0(\exp_{-1}; x) - e^{-x} \geq 0$ and we conclude $\lim_{n\to\infty} a_{1,n} = 0$. In the case $a_{2,n}$, the argumentation is similar and we obtain

$$A_n^0(\exp_{-2}; x) - e^{-2x} \leq \frac{2}{ne}.$$

The theorem is proved. □

Remark 4 The use of the multiplication property makes the proof of the approximation error estimate for the operator A_n^0 in $C_*(\mathbb{R}_0)$ straightforward. As a consequence of Theorem 5, we may write the following estimation

$$\|A_n^0(f) - f\|_\infty \leq 2\,\omega^*\left(f, \frac{2}{\sqrt{en}}\right).$$

In [14] is deduced that for Szász–Mirakyan operators the error of approximation is smaller than the above, so in Section 4, we will investigate the difference of operators $|A_n^0 - S_n|$.

Now, we are going to present a quantitative Voronovskaya-type theorem.

Theorem 6 *If $f, f'' \in C_*(\mathbb{R}_0)$, then we have*

$$\left| n\left[A_n^0(f;x) - f(x)\right] + \frac{1}{2}f'(x) - \frac{x}{2}f''(x)\right|$$
$$\leq \frac{1}{2}\,\omega^*\left(f''; 1/\sqrt{n}\right)\left(n\mu_{n,2}(x) + \left(n^2 A_n^0((e^{-x} - e^{-t})^4; x)\right)^{\frac{1}{2}}\left(n^2\mu_{n,4}(x)\right)^{\frac{1}{2}}\right).$$

Proof Using Taylor's theorem for function f, we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + \gamma(t,x)(t-x)^2, \tag{6}$$

where $\gamma(t,x) = \frac{f''(\alpha) - f''(x)}{2}$ for α between x and t and $\lim_{t\to x}\gamma(t,x) = 0$. Applying operators A_n^0 on both sides of (6) and multiplying by n, we have

$$n\left(A_n^0(f;x) - f(x) - f'(x)A_n^0(e_1 - xe_0; x) - \frac{f''(x)}{2}A_n^0((e_1 - xe_0)^2; x)\right)$$
$$= nA_n^0\Big(\gamma(t,x)(e_1 - xe_0)^2; x\Big).$$

The monotonicity of operators A_n^0 yields

$$\left| n\left(A_n^0(f;x) - f(x) - f'(x)\mu_{n,1}(x) - \frac{f''(x)}{2}\mu_{n,2}(x)\right)\right| \leq nA_n^0\Big(|\gamma(t,x)|(e_1 - xe_0)^2; x\Big). \tag{7}$$

By utilizing the estimation of $\omega^*(f''; \delta)$ (see [14]), we get

$$|\gamma(t,x)| = \left|\frac{f''(\alpha) - f''(x)}{2}\right| \leq \frac{1}{2}\left(1 + \frac{(e^{-\alpha} - e^{-x})^2}{\delta^2}\right)\omega^*(f'', \delta).$$

Using the elementary inequality $(e^{-\alpha} - e^{-x})^2 \leq (e^{-t} - e^{-x})^2$ and the monotonicity of A_n^0 to the estimation above, we get

$$\begin{aligned} A_n^0\Big(|\gamma(t,x)|(e_1 - xe_0)^2; x\Big) &\leq \frac{1}{2}\left[A_n^0\left(\left(1 + \frac{(e^{-t} - e^{-x})^2}{\delta^2}\right)(e_1 - xe_0)^2; x\right)\right]\omega^*(f'', \delta) \\ &= \frac{1}{2}\left[A_n^0((e_1 - xe_0)^2; x) + \frac{A_n^0((e^{-x} - e^{-t})^2(e_1 - xe_0)^2; x)}{\delta^2}\right] \\ &\quad \times \omega^*(f'', \delta) \end{aligned} \tag{8}$$

Now, applying the Cauchy-Schwarz inequality, we have

$$A_n^0\Big((e^{-x} - e^{-t})^2(e_1 - xe_0)^2; x\Big) \leq \left(A_n^0\left((e^{-x} - e^{-t})^4; x\right)\right)^{\frac{1}{2}}\left(A_n^0\left((e_1 - xe_0)^4; x\right)\right)^{\frac{1}{2}}. \tag{9}$$

Combining (8) and (9), we can write

$$\begin{aligned} nA_n^0\Big(|\gamma(t,x)|(e_1 - xe_0)^2; x\Big) &\leq \frac{n}{2}\Bigg(A_n^0((e_1 - xe_0)^2; x) + \frac{1}{\delta^2}\left(A_n^0\left((e^{-x} - e^{-t})^4; x\right)\right)^{\frac{1}{2}} \\ &\quad \times \left(A_n^0\left((e_1 - xe_0)^4; x\right)\right)^{\frac{1}{2}}\Bigg)\omega^*(f'', \delta). \end{aligned}$$

Taking $\delta = \dfrac{1}{\sqrt{n}}$ and using (7), we get the estimation in Theorem 6. □

By Theorem 6 and properties of the modulus of continuity ω^*, we obtain the Voronovskaya-type theorem for operators A_n^0, which can be compared with Theorem 6, in paper [13].

Corollary 1 *Let $f, f'' \in C_*(\mathbb{R}_0)$. Then, for every fixed $x \in \mathbb{R}_0$,*

$$\lim_{n\to\infty} n\Big(A_n^0(f; x) - f(x)\Big) = -\frac{1}{2}f'(x) + \frac{x}{2}f''(x).$$

Proof By Lemma 2, we have

$$\lim_{n\to\infty} n\mu_{n,2}(x) = \lim_{n\to\infty} -2x\,nx\left(\frac{I_1(nx)}{I_0(nx)} - 1\right).$$

Utilizing the following convergence

$$\lim_{z\to\infty} z\left(\frac{I_1(z)}{I_0(z)} - 1\right) = -\frac{1}{2}, \tag{10}$$

we get $\lim_{n\to\infty} n\mu_{n,2}(x) = x$. Furthermore, for $\mu_{n,4}$ we can write

$$\lim_{n\to\infty} n^2\mu_{n,4}(x) = \lim_{n\to\infty}\left(8x^2\left[(nx)^2\left(1-\frac{I_1(nx)}{I_0(nx)}\right)-\frac{1}{2}nx\right]+4x^2nx\left(\frac{I_1(nx)}{I_0(nx)}-1\right)+4x^2\right).$$

Applying (10) and the following limit

$$\lim_{z\to\infty} z\left(z\left(\frac{I_1(z)}{I_0(z)}-1\right)+\frac{1}{2}\right) = -\frac{1}{8}, \tag{11}$$

we get

$$\lim_{n\to\infty} n^2\mu_{n,4}(x) = 3x^2. \tag{12}$$

It remains to prove that $n^2A_n^0((e^{-x}-e^{-t})^4;x)$ is bounded.
Notice that $(e^{-t}-e^{-x})^4 \le (t-x)^4$ for $x \in \mathbb{R}_0$. Therefore, monotonicity of operators A_n^0 yields

$$n^2A_n^0((e^{-t}-e^{-x})^4;x) \le n^2A_n^0((t-x)^4;x) = n^2\mu_{n,4}(x).$$

Taking into account limit (12), Theorem 6 and properties of the modulus of continuity, we conclude the convergence in the corollary. □

Regarding a so-called multiplicative law for linear positive operators usually Grüss–Voronovskaya-type theorem is presented. For operators A_n^0 we have

Theorem 7 *Suppose that $f, g \in C_*^2(\mathbb{R}_0)$. Then for every $x \in \mathbb{R}_0$ we have*

$$\lim_{n\to\infty} n\Big(A_n^0(fg;x) - A_n^0(f;x)A_n^0(g;x)\Big) = xf'(x)g'(x).$$

Proof Let $x \in \mathbb{R}_0$ be fixed and $f, g \in C_*^2(\mathbb{R}_0)$. Consider the following identity

$$A_n^0(fg;x) - A_n^0(f;x)A_n^0(g;x)$$

$$\begin{aligned}
&= \left\{A_n^0(fg;x) - f(x)g(x) - (fg)'(x)\mu_{n,1}(x) - \frac{(fg)''(x)}{2}\mu_{n,2}(x)\right\}\\
&- g(x)\left\{A_n^0(f;x) - f(x) - f'(x)\mu_{n,1}(x) - \frac{f''(x)}{2}\mu_{n,2}(x)\right\}\\
&- A_n^0(f;x)\left\{A_n^0(g;x) - g(x) - g'(x)\mu_{n,1}(x) - \frac{g''(x)}{2}\mu_{n,2}(x)\right\}\\
&+ \frac{1}{2}\mu_{n,2}(x)\left\{f(x)\,g''(x) + 2f'(x)\,g'(x) - g''(x)A_n^0(f;x)\right\}\\
&+ \mu_{n,1}(x)\left\{f(x)\,g'(x) - g'(x)A_n^0(f;x)\right\}
\end{aligned}$$

$$= S_1 - S_2 - S_3 + S_4 + S_5,$$

where

$$\begin{aligned}
S_1 &= A_n^0(fg;x) - f(x)g(x) - (fg)'(x)\mu_{n,1}(x) - \frac{(fg)''(x)}{2}\mu_{n,2}(x),\\
S_2 &= g(x)\left\{A_n^0(f;x) - f(x) - f'(x)\mu_{n,1}(x) - \frac{f''(x)}{2}\mu_{n,2}(x)\right\},\\
S_3 &= A_n^0(f;x)\left\{A_n^0(g;x) - g(x) - g'(x)\mu_{n,1}(x) - \frac{g''(x)}{2}\mu_{n,2}(x)\right\},\\
S_4 &= \frac{1}{2}\mu_{n,2}(x)\left\{f(x)\, g''(x) + 2f'(x)\, g'(x) - g''(x)A_n^0(f;x)\right\},\\
S_5 &= \mu_{n,1}(x)\left\{f(x)\, g'(x) - g'(x)A_n^0(f;x)\right\}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\lim_{n\to\infty} n\Big(A_n^0(fg;x) - A_n^0(f;x)A_n^0(g;x)\Big)\\
&= \lim_{n\to\infty} n(S_1 - S_2 - S_3) + \lim_{n\to\infty} nS_4 + \lim_{n\to\infty} nS_5.
\end{aligned}$$

Utilizing Theorem 6 for $f, g, fg \in C_*^2(\mathbb{R}_0)$, we get

$$\lim_{n\to\infty} n(S_1 - S_2 - S_3) = 0.$$

Moreover, using Lemma 2, (10) and Theorem 5, we obtain the following convergence

$$\begin{aligned}
\lim_{n\to\infty} nS_4 &= \lim_{n\to\infty} \frac{n}{2} g''(x)\mu_{n,2}(x)\Big(f(x) - A_n^0(f;x)\Big) + f'(x)g'(x)\lim_{n\to\infty} n\mu_{n,2}(x)\\
&= xf'(x)g'(x)
\end{aligned}$$

and

$$\lim_{n\to\infty} nS_5 = \lim_{n\to\infty} ng'(x)\mu_{n,1}(x)\Big(f(x) - A_n^0(f;x)\Big) = 0.$$

Taking into account the above relations we have the required result. □

As expected, the multiplicative law concerning operators A_n^0 is not true. However, as a consequence of Theorem 7, we have $A_n^0(fg;x) - A_n^0(f;x)A_n^0(g;x) = O\,(1/n)$ as $n \to \infty$.

4 Difference of operators

In this section we will consider the difference between two operators, that means we estimate difference $|A_n^0 - S_n|$. The investigation of differences between operators is

an ongoing problem in operator theory, see for example [1], [8], [10].
The main tool, we shall apply, is Theorem 2.1 from paper [8] and results from [12].
For our operators we define

$$s_{n,j}(x) = e^{-nx}\frac{(nx)^j}{j!}, \quad H_{n,j}(f) = f\left(\frac{j}{n}\right)$$

and

$$a_{n,j}(x) = \frac{1}{I_0(nx)}\frac{(\frac{nx}{2})^{2j}}{(\Gamma(j+1))^2}, \quad J_{n,j}(f) = f\left(\frac{2j}{n}\right).$$

Therefore, operators S_n and A_n^{ν} can be represented as

$$S_n(f;x) = \sum_{j=0}^{\infty} s_{n,j}(x)H_{n,j}(f), \quad A_n^0(f;x) = \sum_{j=0}^{\infty} a_{n,j}(x)J_{n,j}(f).$$

Moreover, we denote by

$$b^{H_{n,j}} = H_{n,j}(e_1), \quad \mu_2^{H_{n,j}} = H_{n,j}(e_1 - b^{H_{n,j}}e_0),$$

and

$$b^{J_{n,j}} = J_{n,j}(e_1), \quad \mu_2^{J_{n,j}} = J_{n,j}(e_1 - b^{J_{n,j}}e_0).$$

Remark 5 Based on paper [8] we can write for S_n

$$b^{H_{n,j}} = \frac{j}{n}, \quad \mu_2^{H_{n,j}} = 0.$$

Remark 6 For operators A_n^0 we can compute similar quantities. Using the definition of $b^{J_{n,j}}$, $\mu_2^{J_{n,j}}$ we get

$$b^{J_{n,j}} = \frac{2j}{n}, \quad \mu_2^{J_{n,j}} = 0.$$

Now, we are in a position to formulate the theorem.

Theorem 8 *If $f \in C_b(\mathbb{R}_0)$ and $x \in \mathbb{R}_0$, then for $n \in \mathbb{N}$ we have*

$$|(S_n - A_n^0)(f;x)| \le 2\omega_1\left(f;\sqrt{\frac{x}{n}}\right) + 2\omega_1\left(f;\sqrt{2x^2\left(1-\frac{I_1(nx)}{I_0(nx)}\right)}\right).$$

Proof By aforemention theorem we have

$$|(S_n - A_n^0)(f;x)| \leq \alpha(x)\|f''\| + 2\omega_1(f;\delta_1(x)) + 2\omega_1(f;\delta_2(x))$$

where

$$\alpha(x) = \frac{1}{2}\sum_{j=0}^{\infty}\left(s_{n,j}(x)\mu_2^{H_{n,j}} + a_{n,j}(x)\mu_2^{J_{n,j}}\right),$$
$$\delta_1^2(x) = \sum_{j=0}^{\infty} s_{n,j}(x)\left(b^{H_{n,j}} - x\right)^2,$$
$$\delta_2^2(x) = \sum_{j=0}^{\infty} a_{n,j}(x)\left(b^{J_{n,j}} - x\right)^2.$$

By definitions of $s_{n,j}(x)$ and $a_{n,j}(x)$, using Remark 4 and Remark 5 it is easy to observe that $\alpha(x) = 0$. For $\delta_1^2(x)$ we get

$$\delta_1^2(x) = \sum_{j=0}^{\infty} s_{n,j}(x)\left(\frac{j}{n} - x\right)^2 = \frac{x}{n},$$

and for $\delta_2^2(x)$ we achieve

$$\delta_2^2(x) = \sum_{j=0}^{\infty} a_{n,j}\left(\frac{2j}{n} - x\right)^2 = 2x^2\left(1 - \frac{I_1(nx)}{I_0(nx)}\right).$$

The theorem is proved. □

5 Graphical representation

In this section we use the Mathematica software to visualize the convergence of our operators. For $x \in [0,5] \subset \mathbb{R}_0$ we deal with the function $f(x) = \frac{1}{1+x+x^2}$ which belongs to the space $C_*(\mathbb{R}_0)$. Figure 1 performs four terms of the sequence of operators A_n^0, for $n = 5, 10, 20, 50$. (See Fig. 1).

The graphs ensure as that for any compact subset of $\mathbb{R}_0$ operators A_n^0 are uniformly convergent to the approximated function $f(x) = \frac{1}{1+x+x^2}$, which was proved in Theorem 3.

In the picture below (see Fig. 2), we enlarge the plots that we have above for $x \in [3,5]$.

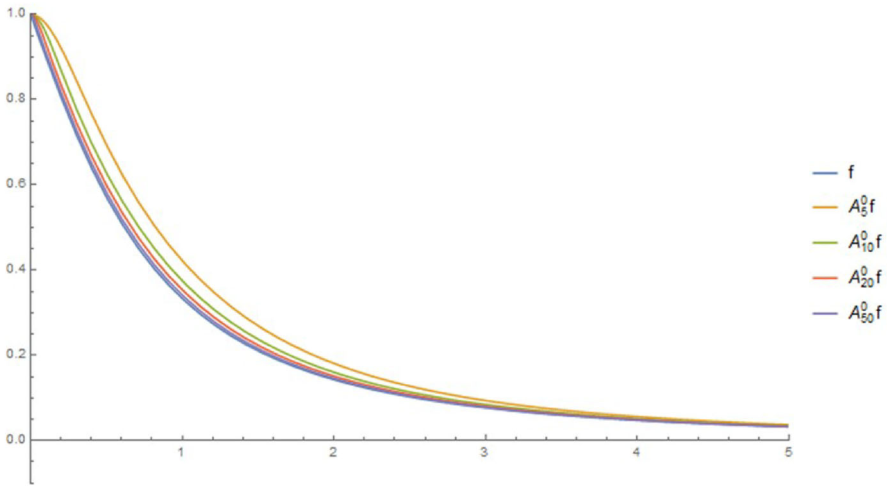

Fig. 1 From the top $A^0_5(f)$, $A^0_{10}(f)$, $A^0_{20}(f)$, $A^0_{50}(f)$, f.

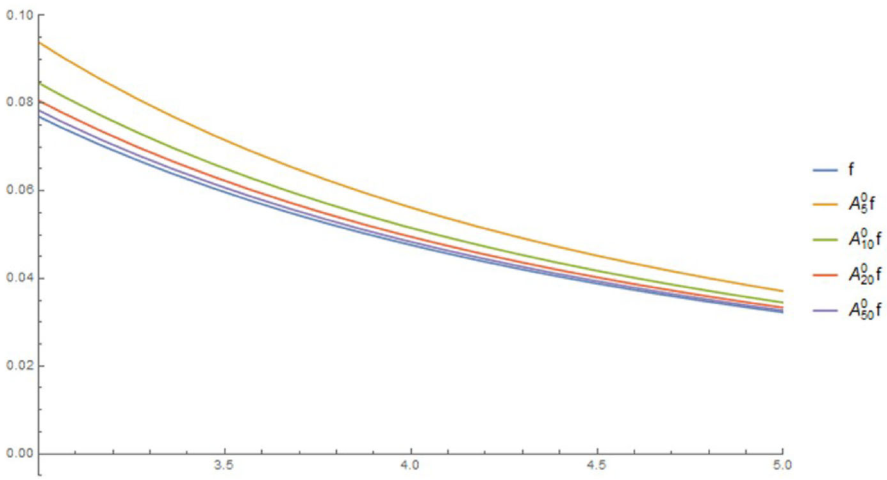

Fig. 2 From the top $A^0_5(f)$, $A^0_{10}(f)$, $A^0_{20}(f)$, $A^0_{50}(f)$, f.

6 Conclusion

In the article we focus on a special case of operators introduced in paper [12]. We propose a new approach to the operators regarding new achievements in the area of operator research. We take into account the operators because of their good approximation properties, considering their error of approximation for uniformly continuous functions. It is proved in paper [15] that the second central moment for these operators is smaller then the genuine Szász–Mirakyan ones. Hence, we investigate the error of convergence for the operators in space $C_*(\mathbb{R}_0)$, utilizing an interesting multiplication relation for modified Bessel functions. Furthermore, we achieve the Voronovskaya-type results due to new asymptotic formulas for functions I_1 and I_0. We conclude these considerations presenting the difference between A_n^0 and basic Szász–Mirakyan operators. In view of great attention which attract Bessel functions (see for example papers [20], [21], [23]) we propose to consider a generalized form of our operators for various scaling.

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