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Some remarks about the convergence of composition operators

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Abstract

This article presents a new sequence of positive linear operators that results from recent advancements in the area of composition operators. We introduce the moment-generating function and utilize it to compute moments of certain orders. Regarding the convergence, we establish fundamental approximation theorems as well as Voronovskaya-type and Grüss-Voronovskaya-type theorems. Moreover, the difference estimates of the Baskakov-Kantorovich-type and our newly defined composition operators are also provided.

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1 Introduction

Let us denote the space of all real-valued continuous functions on $\mathbb{R}_0 := [0,\infty)$ by $C(\mathbb{R}_0)$, and the Banach space of all continuous and bounded functions on $\mathbb{R}_0$ endowed with the sup-norm $\|\cdot\|_\infty$ by $C_b(\mathbb{R}_0)$. Moreover, we shall also consider the following closed subspace of $C_b(\mathbb{R}_0)$:

$$C_*(\mathbb{R}_0) := \left\{f \in C_b(\mathbb{R}_0) \mid \lim_{x\to\infty} f(x) = g\right\},$$

where $g \in \mathbb{R}$.

Remark 1 As usual, we denote $e_n(t) = t^n$ the monomial of degree $n \in \mathbb{N}$.

Remark 2 For improved presentation and readability of the paper, we use both e^x and $\exp(x)$ notations for the exponential function e^x, where appropriate. Moreover, we utilize the software Mathematica to do some calculations.

Whenever we investigate approximation operators, we are interested in the convergence of these operators to a given function f. One of the possible answers to the question presents Theorem 2.1 from paper [4].

Theorem 1 (*Theorem* 2.1, [4]) *Suppose that for each* $s \in \mathbb{R}$ *and* $x \in \mathbb{R}_0$,

$$\lim_{n\to\infty} L_n(e^{ist};x) = L(e^{ist};x),$$

then

$$\lim_{n\to\infty} L_n(f(t);x) = L(f(t);x),$$

for all $f \in C_b(\mathbb{R}_0)$.

We also recall Theorem 2.1 from paper [16] to get an estimation of the error of approximation.

Theorem 2 (*Theorem* 2.1, [16]) *Let the sequence of positive linear operators*

$$L_n : C_*[0,\infty) \to C_*[0,\infty),$$

satisfies the following conditions

$$\lim_{n\to\infty} a_{m,n} = 0, \quad m \in \{0,1,2\},$$

where

$$a_{m,n} := \|L_n(\exp_{-m}) - \exp_{-m}\|_\infty.$$

Then for all $f \in C_*[0,\infty)$, *we have*

$$\|L_n(f) - f\|_\infty \le a_{0,n}\|f\|_\infty + (2 + a_{0,n})\,\omega^*\left(f, \sqrt{a_{0,n} + 2a_{1,n} + a_{2,n}}\right),$$

where

$$\omega^*(f,\delta) := \sup\left\{|f(x) - f(t)| : x,t \ge 0, |e^{-x} - e^{-t}| \le \delta\right\}.$$

Remark 3 [16] For $f \in C_*[0,\infty)$ the modulus of continuity $\omega^*(f,\delta)$ has the following connections with the classical modulus of continuity $\omega(f,\delta)$:

(i) $\omega^*(f,\delta) = \omega(f^*,\delta)$, where $f^* : [0,1] \to \mathbb{R}$ is given by

$$f^*(x) = \begin{cases} f(-\ln\, x), & \text{if } x \in (0,1] \\ \lim_{t\to\infty} f(t), & \text{if } x = 0, \end{cases}$$

(ii) for every $\delta \ge 0$ we have $\omega(f,\delta) \le \omega^*(f,\delta)$.

In the past few decades, developing new operators has consistently intrigued academics. Several techniques have been documented in literature for this purpose including the composition of operators, see for example: [5–8, 18–20] and [21]. Recently, Gupta [10] used composition technique and modified Laguerre polynomials to construct new approximation operators. He discovered discrete type operators and presented their explicit representation, moment generating functions and convergence theorems. Abel and Gupta [1], performed composition of well known operators like Bernstein–Durrmeyer and Bernstein, Szász–Durrmeyer and Jain–Pethe, Baskakov–Szász and Szász–Mirakyan operators to introduce new operators which approximate real continuous functions. In this direction, Acu et al. [3] established the composition of semi-exponential Szász-Mirakyan operators and Szász–Durrmeyer operators with Post–Widder operators. For more information on composition operators, we suggest the papers [2, 11, 14].

2 Composition of semi-exponential Post–Widder operators and Szász–Kantorovich operators

We start this section by short preliminary recalling basic definitions connected with our new operators.

Szász–Kantorovich and Baskakov–Kantorovich operators [13] are defined as follows:

$$\overline{S}_n(f;x) = ne^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt,$$

$$\overline{V}_n(f;x) = n\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt,$$

for $n\in\mathbb{N}, x\in\mathbb{R}_0$ and $f\in C_b(\mathbb{R}_0)$. Semi-exponential Szász–Mirakyan and semi-exponential Post–Widder operators are defined

$$S_n^{\beta}(f;x) = e^{-(n+\beta)x}\sum_{k=0}^{\infty}\frac{((n+\beta)x)^k}{k!}f\left(\frac{k}{n}\right)$$

$$P_n^{\beta}(f;x) = \frac{1}{e^{\beta x}}\left(\frac{n}{x}\right)^n\sum_{k=0}^{\infty}\frac{(n\beta)^k}{k!}\frac{1}{\Gamma(n+k)}\int_0^{\infty} e^{-\frac{nu}{x}}u^{n+k-1}f(u)du$$

for $\beta>0$, $n\in\mathbb{N}$, $x\in\mathbb{R}_0$ and $f\in C_b(\mathbb{R}_0)$.

In paper [3] composition operators $L_n^{\beta}(.;x)$ are defined as follows:

$$L_n^{\beta}(.;x) := (P_n\circ S_n^{\beta})(.;x),$$

where $P_n(.;x)$ is the classical Post–Widder operator. The explicit formula of the operator is

$$L_n^{\beta}(f(t);x) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(n)}\left(\frac{n}{x}\right)^n\int_0^{\infty} e^{-\frac{nu}{x}}u^{n-1}e^{-(n+\beta)u}\frac{((n+\beta)u)^k}{k!}f\left(\frac{k}{n}\right)du,$$

which we also can rewrite as Baskakov-type operators $V_n(.;x)$

$$L_n^{\beta}(f;x) = V_n\left(f;\frac{(n+\beta)x}{n}\right).$$

In this article, we will consider the following operators:

$$K_n^{\beta}(.;x) := (P_n^{\beta}\circ\overline{S}_n)(.;x),$$

which can be written as

$$K_n^{\beta}(f(t);x) = ne^{-\beta x}\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\binom{n+m-1}{j}\frac{\beta^{m-j}}{(m-j)!}\frac{x^m}{(1+x)^{m+n}}\int_{\frac{j}{n}}^{\frac{j+1}{n}} f(t)dt.$$

We start from setting the MGF (Moment Generating Function) for our operators.

Proposition 1 *For $\beta > 0$ and $x \in \mathbb{R}_0$, the moment generating function of $K_n^{\beta}(.;x)$ is given by*

$$K_n^{\beta}(\exp_A;x) = \frac{n}{A}\left(e^{\frac{A}{n}}-1\right)\left(1-x\left(e^{\frac{A}{n}}-1\right)\right)^{-n}\exp\left(\frac{\beta\left(e^{\frac{A}{n}}-1\right)x^2}{1-\left(e^{\frac{A}{n}}-1\right)x}\right),$$

for A in some open neighborhood of 0.

Also, if $A = 0$ then we will consider the limit $\lim_{A\to 0}\frac{n}{A}\left(e^{\frac{A}{n}}-1\right) = 1$.

Proof By the definition of the operators $K_n^{\beta}(.;x)$, we have

$$K_n^{\beta}(\exp_A;x) = P_n^{\beta}\circ\overline{S}_n(\exp_A;x) = P_n^{\beta}(\overline{S}_n\exp_A;x) = \frac{n}{A}\left(e^{\frac{A}{n}}-1\right)P_n^{\beta}\left(\exp_{n\left(e^{\frac{A}{n}}-1\right)};x\right).$$

Now, applying the moment generating function of the operator $P_n^{\beta}(.;x)$ (see [3]), we get the desired result. □

We obtain the moments of our operators using Proposition 1 and the following well known connection:

$$K_n^{\beta}(e_r;x) = \left[\frac{\partial^r}{\partial A^r}K_n^{\beta}(\exp_A;x)\right]_{A=0}.$$

Lemma 1 *Let $\beta > 0$ and $x \in \mathbb{R}_0$. Then the following formulas hold true*:

(i) $K_n^{\beta}(e_0;x) = 1$;

(ii) $K_n^{\beta}(e_1;x) = x + \frac{2\beta x^2+1}{2n}$;

(iii) $K_n^{\beta}(e_2;x) = x^2 + \frac{2\beta x^3+x^2+2x}{n} + \frac{3\beta^2x^4+6\beta x^3+6\beta x^2+1}{3n^2}$;

(iv) $$K_n^{\beta}(e_3;x) = x^3 + \frac{6\beta x^4+6x^3+9x^2}{2n} + \frac{6\beta^2x^5+18\beta x^4+18\beta x^3+4x^3+9x^2+7x}{2n^2} + \frac{4\beta^3x^6+24\beta^2x^5+18\beta^2x^4+24\beta x^4+36\beta x^3+14\beta x^2+1}{4n^3};$$

(v) $$\begin{aligned}K_n^{\beta}(e_4;x) = x^4 &+ \frac{1}{5n^4}\left[5\beta^4x^8+(20\beta^3n+60\beta^3)x^7\right.\\ &\left.+(30\beta^2n^2+150\beta^2n+40\beta^3+180\beta^2)x^6\right]\\ &+\frac{1}{5n^4}\left[(20\beta n^3+120\beta n^2+(120\beta^2+220\beta)n+240\beta^2+120\beta)x^5\right]\\ &+\frac{1}{5n^4}\left[(30n^3+(120\beta+55)n^2+(360\beta+30)n+75\beta^2+240\beta)x^4\right]\\ &+\frac{1}{5n^4}\left[(40n^3+120n^2+(150\beta+180)n+150\beta)x^3\right.\\ &\left.+(75n^2+75n+30\beta)x^2+30nx+1\right].\end{aligned}$$

If we denote by $\mu_{n,r}^{\beta}(x) = K_n^{\beta}\left((e_1 - x)^r; x\right) = K_n^{\beta}\left((e_1 - xe_0)^r; x\right)$, the $r^{th} (r \in \mathbb{N})$ order central moments of the operators $K_n^{\beta}(.;x)$, then we can compute $\mu_{n,r}^{\beta}(x)$ with the help of the following relation:

$$\mu_{n,r}^{\beta}(x) = \left[\frac{\partial^r}{\partial A^r}\left\{\frac{n}{A}\left(e^{\frac{A}{n}}-1\right)\left(1-x\left(e^{\frac{A}{n}}-1\right)\right)^{-n}\exp\left(\frac{\beta\left(e^{\frac{A}{n}}-1\right)x^2}{1-\left(e^{\frac{A}{n}}-1\right)x}-Ax\right)\right\}\right]_{A=0}.$$

Lemma 2 *For given $\beta > 0$ and $x \in \mathbb{R}_0$, the operators $K_n^{\beta}(.;x)$ verify the following identities*:

(i) $\mu_{n,0}^{\beta}(x) = 1$;

(ii) $\mu_{n,1}^{\beta}(x) = \dfrac{2\beta x^2 + 1}{2n}$;

(iii) $\mu_{n,2}^{\beta}(x) = \dfrac{x(x+1)}{n} + \dfrac{6\beta^2x^4 + 12\beta x^3 + 12\beta x^2 + 2}{6n^2}$;

(iv) $$\mu_{n,3}^{\beta}(x) = \frac{1}{4n^3}\left[4\beta^3x^6 + 24\beta^2x^5 + (12\beta n + 18\beta^2 + 24\beta)x^4\right] + \frac{1}{4n^3}\left[((12\beta + 8)n + 36\beta)x^3 + (18n + 14\beta)x^2 + 10nx + 1\right];$$

(v) $$\begin{aligned}\mu_{n,4}^{\beta}(x) &= \frac{1}{5n^4}\left[5\beta^4x^8 + 60\beta^3x^7 + (30\beta^2n + 40\beta^3 + 180\beta^2)x^6\right] \\ &+ \frac{1}{5n^4}\left[((30\beta^2 + 100\beta)n + 240\beta^2 + 120\beta)x^5\right] \\ &+ \frac{1}{5n^4}\left[(15n^2 + (180\beta + 30)n + 75\beta^2 + 240\beta)x^4\right] \\ &+ \frac{1}{5n^4}\left[(30n^2 + (80\beta + 80)n + 150\beta)x^3 + (15n^2 + 75n + 30\beta)x^2 + 25nx + 1\right].\end{aligned}$$

The next theorem is a consequence of Theorem 1. First, we achieve the convergence of our operators to a given function f. Next, we regard different subsequences of $K_n^{\beta}(.;x)$ and we get, as the limits, well-know operators.

Theorem 3 *Let $f \in C_b(\mathbb{R}_0)$, $x \in \mathbb{R}_0$ and m be a non-negative integer. Then the following convergence holds true*:

(i) $\lim_{n\to\infty} K_n^{\beta}\left(f(t);x\right) = f(x)$,

(ii) $\lim_{n\to\infty} K_{mn}^{\beta}\left(f(nt); \dfrac{x}{n}\right) = \overline{S}_m\left(f(t);x\right)$,

(iii) $\lim_{\beta\to 0} K_n^{\beta}\left(f(t);x\right) = \overline{V}_n\left(f(t);x\right)$,

(iv) $\lim_{n\to\infty} K_n^{\beta}\left(f\left(\dfrac{t}{n}\right); nx\right) = f(\beta x^2 - x)$, *for* $x \geq \dfrac{1}{\beta}$.

Proof

(i) By the definition of the operator $K_n^{\beta}(.;x)$ and the MGF of this operator we have

$$K_n^{\beta}(e^{ist};x) = \frac{n}{is}\left[\exp\left(\frac{is}{n}\right)-1\right]\left(1-x\left[\exp\left(\frac{is}{n}\right)-1\right]\right)^{-n}\exp\left(\frac{\beta(x)^2\left[\exp\left(\frac{is}{n}\right)-1\right]}{1-x\left[\exp\left(\frac{is}{n}\right)-1\right]}\right),$$

and

$$\lim_{n\to\infty} K_n^{\beta}(e^{ist};x) = e^{isx} = Id(e^{ist};x).$$

Hence, by Theorem 1, we get the convergence of our operators $K_n^{\beta}(.;x)$ to the approximated function f.

(ii) Consider the expression

$$K_{mn}^{\beta}\left(e^{isnt};\frac{x}{n}\right)=\frac{m}{is}\left[\exp\left(\frac{is}{m}\right)-1\right]\left(1-\frac{x}{n}\left[\exp\left(\frac{is}{m}\right)-1\right]\right)^{-mn}$$
$$\times\exp\left(\frac{\beta\left(\frac{x}{n}\right)^2\left[\exp\left(\frac{is}{m}\right)-1\right]}{1-\frac{x}{n}\left[\exp\left(\frac{is}{m}\right)-1\right]}\right).$$

Notice that the third factor of the above expression tends to $\exp\left(mx\left[\exp\left(\frac{is}{m}\right)-1\right]\right)$ as $n\to\infty$ and the last one tends to 1 as $n\to\infty$. So we get the limit

$$\lim_{n\to\infty}K_{mn}^{\beta}\left(e^{isnt};\frac{x}{n}\right)=\frac{m}{is}\left[\exp\left(\frac{is}{m}\right)-1\right]\exp\left(mx\left[e^{\frac{is}{m}}-1\right]\right)=\overline{S}_m(e^{ist};x).$$

In virtue of Theorem 1, we get part (ii) in the theses of our theorem.

(iii) To prove part (iii), we have to write the following expression

$$K_n^{\beta}(e^{ist};x)=\frac{n}{is}\left[\exp\left(\frac{is}{n}\right)-1\right]\left(1-x\left[\exp\left(\frac{is}{n}\right)-1\right]\right)^{-n}\exp\frac{\beta x^2\left[\exp\left(\frac{is}{n}\right)-1\right]}{1-x\left[\exp\left(\frac{is}{n}\right)-1\right]}.$$

It is easy to observe that

$$\lim_{\beta\to 0}K_n^{\beta}(e^{ist};x)=\frac{n}{is}\left[\exp\left(\frac{is}{n}\right)-1\right]\left(1-x\left[\exp\left(\frac{is}{n}\right)-1\right]\right)^{-n}=\overline{V}_n(e^{ist};x).$$

Theorem 1 leads to the required equality.

(iv) Let us consider $K_n^{\beta}(e^{\frac{ist}{n}};nx)$. Applying Proposition 1, we have

$$K_n^{\beta}(e^{\frac{ist}{n}};nx)=\frac{n^2}{is}\left[\exp\left(\frac{is}{n^2}\right)-1\right]\left(1-nx\left[\exp\left(\frac{is}{n^2}\right)-1\right]\right)^{-n}$$
$$\times\exp\left(\frac{\beta\,(nx)^2\left[\exp\left(\frac{is}{n^2}\right)-1\right]}{1-nx\left[\exp\left(\frac{is}{n^2}\right)-1\right]}\right).$$

Therefore, we get

$$\lim_{n\to\infty}K_n^{\beta}(e^{\frac{ist}{n}};nx)=e^{is(\beta x^2-x)}.$$

If we define the operator $L(f(t);x)=f(\beta x^2-x)$, for $x\geq\dfrac{1}{\beta}$, we can deduce that

$$\lim_{n\to\infty}K_n^{\beta}\left(f\left(\frac{t}{n}\right);nx\right)=f(\beta x^2-x).$$

Again, we apply Theorem 1 for our operators, and we get the last limit stated in theorem's conclusion. □

Now, we move onto the estimation of the error of approximation.

Theorem 4 *Let $f \in C_*(\mathbb{R}_0)$, then we have*

$$\|K_n^\beta(f) - f\|_\infty \le 2\,\omega^*\left(f, \sqrt{2a_{1,n} + a_{2,n}}\right),$$

where

$$\lim_{n\to\infty} a_{1,n} = \lim_{n\to\infty} \|K_n^\beta(\exp_{-1}) - \exp_{-1}\|_\infty = 0,$$

and

$$\lim_{n\to\infty} a_{2,n} = \lim_{n\to\infty} \|K_n^\beta(\exp_{-2}) - \exp_{-2}\|_\infty = 0.$$

Proof By Lemma 1, we notice that operators $K_n^\beta(.;x)$ preserve constants, therefore in Theorem 2 coefficient $a_{0,n} = 0$. Next, we exam the limit of $a_{1,n}$ and $a_{2,n}$ as $n \to \infty$.

The MGF of the operators yields

$$K_n^\beta(\exp_{-1};x) = -n(e^{-\frac{1}{n}} - 1)\left(1 - x(e^{-\frac{1}{n}} - 1)\right)^{-n} \exp\left(\frac{\beta\left(e^{-\frac{1}{n}} - 1\right)x^2}{1 - \left(e^{-\frac{1}{n}} - 1\right)x}\right),$$

and

$$K_n^\beta(\exp_{-2};x) = -\frac{n}{2}(e^{-\frac{2}{n}} - 1)\left(1 - x(e^{-\frac{2}{n}} - 1)\right)^{-n} \exp\left(\frac{\beta\left(e^{-\frac{2}{n}} - 1\right)x^2}{1 - \left(e^{-\frac{2}{n}} - 1\right)x}\right).$$

By Theorem 3 part (i), for $x \in \mathbb{R}_0$, we immediately get

$$\lim_{n\to\infty} K_n^\beta(\exp_{-1};x) = e^{-x},$$

and

$$\lim_{n\to\infty} K_n^\beta(\exp_{-2};x) = e^{-2x}.$$

Let us substitute $F_n = e^{-\frac{1}{n}} - 1$ and $G_n = e^{-\frac{2}{n}} - 1$, then we have

$$\begin{aligned} K_n^\beta(\exp_{-1};x) &= -nF_n(1 - xF_n)^{-n} \exp\left(\frac{\beta x^2 F_n}{1 - xF_n}\right) \\ &= -nF_n \exp\left(\frac{\beta x^2 F_n}{1 - xF_n} - n\ln(1 - xF_n)\right), \end{aligned}$$

and

$$\begin{aligned} K_n^\beta(\exp_{-2};x) &= -\frac{nG_n}{2}(1 - xG_n)^{-n} \exp\left(\frac{\beta x^2 G_n}{1 - xG_n}\right) \\ &= -\frac{nG_n}{2} \exp\left(\frac{\beta x^2 G_n}{1 - xG_n} - n\ln(1 - xG_n)\right). \end{aligned}$$

Now, for $n \in \mathbb{N}$ and $0 \leq x \leq -\frac{1}{G_n} < -\frac{1}{F_n}$, we consider two sequences: f_n, g_n as:

$$f_n(x) = K_n^{\beta}(\exp_{-1}; x) - e^{-x}, \quad g_n(x) = K_n^{\beta}(\exp_{-2}; x) - e^{-2x}.$$

Using Taylor's expansion of $\frac{\beta x^2 F_n}{1 - xF_n} - n\ln(1 - xF_n)$ and $\frac{\beta x^2 G_n}{1 - xG_n} - n\ln(1 - xG_n)$, we can write, respectively

$$\begin{aligned} f_n(x) &= -nF_n \exp\left(\frac{\beta x^2 F_n}{1 - xF_n} - n\ln(1 - xF_n)\right) - e^{-x} \\ &= -nF_n \exp\left(F_n nx + \frac{(F_n^2 n + 2F_n b)x^2}{2} + \frac{(F_n^3 n + 3F_n^2 b)x^3}{3}\right) \\ &\quad \times \exp\left(\frac{(F_n^4 n + 4F_n^3 b)x^4}{4} + \frac{(F_n^5 n + 5F_n^4 b)x^5}{5} + \cdots\right) - e^{-x}, \end{aligned}$$

and

$$\begin{aligned} g_n(x) &= -\frac{nG_n}{2} \exp\left(\frac{\beta x^2 G_n}{1 - xG_n} - n\ln(1 - xG_n)\right) - e^{-2x} \\ &= -\frac{nG_n}{2} \exp\left(G_n nx + \frac{(G_n^2 n + 2G_n b)x^2}{2} + \frac{(G_n^3 n + 3G_n^2 b)x^3}{3}\right) \\ &\quad \times \exp\left(\frac{(G_n^4 n + 4G_n^3 b)x^4}{4} + \cdots\right) - e^{-2x}. \end{aligned}$$

Since $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} g_n(0) = 0$ and $\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} g_n(x) = 0$, there exist points ξ_n, ρ_n such that

$$\|f_n\|_\infty = f_n(\xi_n) \quad \text{and} \quad \|g_n\|_\infty = g_n(\rho_n),$$

and

$$f_n'(\xi_n) = g_n'(\rho_n) = 0. \tag{2.1}$$

Computing the derivatives of f_n and g_n, we get

$$\begin{aligned} f_n'(x) &= e^{-x} - nF_n \exp\left(\frac{\beta x^2 F_n}{1 - xF_n} - n\ln(1 - xF_n)\right) \\ &\quad \times (F_n n + (F_n^2 n + 2F_n\beta)x + (F_n^3 n + 3F_n^2\beta)x^2 + (F_n^4 n + 4F_n^2\beta)x^3 + \cdots), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} g_n'(x) &= 2e^{-2x} - \frac{nG_n}{2} \exp\left(\frac{\beta x^2 G_n}{1 - xG_n} - n\ln(1 - xG_n)\right) \\ &\quad \times (G_n n + (G_n^2 n + 2G_n\beta)x + (G_n^3 n + 3G_n^2\beta)x^2 + (G_n^4 n + 4G_n^2\beta)x^3 + \cdots). \end{aligned} \tag{2.3}$$

Applying the definitions of f_n, g_n and connections (2.1)-(2.3), we have

$$f_n(\xi_n) = -nF_n \exp\left(\frac{\beta(\xi_n)^2 F_n}{1 - \xi_n F_n} - n\ln(1 - \xi_n F_n)\right)$$

$$\times (1 + F_n n + (F_n^2 n + 2F_n\beta)\xi_n + (F_n^3 n + 3F_n^2\beta)(\xi_n)^2 + (F_n^4 n + 4F_n^2\beta)(\xi_n)^3 + \cdots).$$

By elementary calculations, we notice that

$$\lim_{n\to\infty} F_n = 0, \quad \lim_{n\to\infty} F_n n = -1.$$

As a consequence $\lim_{n\to\infty} f_n(\xi_n) = 0$. Analogously for $g_n(\rho_n)$, we have

$$g_n(\rho_n) = -\frac{nG_n}{2}\frac{1}{2}\exp\left(\frac{\beta(\rho_n)^2 G_n}{1-\rho_n G_n} - n\ln(1-\rho_n G_n)\right)$$
$$\times (2 + G_n n + (G_n^2 n + 2G_n\beta)\rho_n + (G_n^3 n + 3G_n^2\beta)(\rho_n)^2$$
$$+ (G_n^4 n + 4G_n^2\beta)(\rho_n)^3 + \cdots),$$

and

$$\lim_{n\to\infty} G_n = 0, \quad \lim_{n\to\infty} G_n n = -2.$$

Hence, we have $\lim_{n\to\infty} g_n(\rho_n) = 0$. The theorem is proved. □

3 Voronovskaya-type theorems

The basic tools for investigating the point-wise convergence of positive linear operators are Voronovskaya-type results. We consider the theorems for $f \in C_*(\mathbb{R}_0)$ and the appropriate modulus of continuity $\omega^*(f,\delta)$.

Theorem 5 *Consider the function $f \in C_*(\mathbb{R}_0)$ such that $f', f'' \in C_*(\mathbb{R}_0)$. Then, for $x \in \mathbb{R}_0$, the operators $K_n^\beta(f;x)$ admit the following relation*:

$$\left|2n\left(K_n^\beta(f;x) - f(x) - f'(x)\mu_{n,1}^\beta(x) - \frac{f''(x)}{2}\mu_{n,2}^\beta(x)\right)\right|$$
$$\leq \omega^*\left(f'', \frac{1}{\sqrt{n}}\right)\left(n\mu_{n,1}^\beta(x) + \left(n^2 K_n^\beta\left((e^{-x}-e^{-t})^4;x\right)\right)^{\frac{1}{2}}\left(n^2\mu_{n,4}^\beta(x)\right)^{\frac{1}{2}}\right). \tag{3.1}$$

Proof Using Taylor's formula for the function f, we can write

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + \gamma_2(t,x)(t-x)^2, \tag{3.2}$$

where

$$\gamma_2(t,x) = \frac{f''(\alpha) - f''(x)}{2}, \tag{3.3}$$

and without loss of generality, we assume $\alpha \in (x,t);\ x,\ t \in \mathbb{R}_0$.

Applying the operators $K_n^\beta(.;x)$ on both sides of (3.2), we have

$$\left|2n\left(K_n^\beta(f;x) - f(x) - f'(x)\mu_{n,1}^\beta(x) - \frac{f''(x)}{2}\mu_{n,2}^\beta(x)\right)\right|$$
$$\leq 2nK_n^\beta\left(|\gamma_2(t,x)|(e_1-x)^2;x\right). \tag{3.4}$$

By utilizing the estimation of $\omega^*(f'';\delta)$ (see [16]), we get

$$|\gamma_2(t,x)| = \left|\frac{f''(\alpha)-f''(x)}{2}\right| \leq \frac{1}{2}\left(1+\frac{(e^{-x}-e^{-t})^2}{\delta^2}\right)\omega^*(f'',\delta). \tag{3.5}$$

Using inequality (3.5), we obtain

$$K_n^\beta\left(|\gamma_2(t,x)|(e_1-x)^2;x\right) \leq \frac{1}{2}\left(K_n^\beta((e_1-x)^2;x)+\frac{K_n^\beta((e^{-x}-e^{-t})^2(e_1-x)^2;x)}{\delta^2}\right). \times \omega^*(f'',\delta). \tag{3.6}$$

But then from Cauchy-Schwarz inequality, we have

$$K_n^\beta\left((e^{-x}-e^{-t})^2(e_1-x)^2;x\right) \leq \left(K_n^\beta\left((e^{-x}-e^{-t})^4;x\right)\right)^{\frac{1}{2}}\left(K_n^\beta\left((e_1-x)^4;x\right)\right)^{\frac{1}{2}}. \tag{3.7}$$

Combining (3.6) and (3.7), we can write

$$2nK_n^\beta\left(|\gamma_2(t,x)|(e_1-x)^2;x\right) \leq n\left(K_n^\beta((e_1-x)^2;x)+\frac{1}{\delta^2}\left(K_n^\beta\left((e^{-x}-e^{-t})^4;x\right)\right)^{\frac{1}{2}} \times \left(K_n^\beta\left((e_1-x)^4;x\right)\right)^{\frac{1}{2}}\right)\omega^*(f'',\delta). \tag{3.8}$$

Now, by taking $\delta = \frac{1}{\sqrt{n}}$ and using (3.4), we reach at the proof. □

As a corollary, we get the Voronovskaya theorem for operators $K_n^\beta(.;x)$.

Corollary 1 *Assume that $f \in C_*(\mathbb{R}_0)$ such that $f', f'' \in C_*(\mathbb{R}_0)$. Then, for any $x \in \mathbb{R}_0$,*

$$\lim_{2n\to\infty} n\left(K_n^\beta(f;x)-f(x)\right) = (2\beta x^2+1)f'(x)+2x(x+1)f''(x).$$

Proof In view of the linear property and the MGF of the operators $K_n^\beta(.;x)$, we obtain

$$\begin{aligned}\lim_{n\to\infty} n^2K_n^\beta\left((e^{-x}-e^{-t})^4;x\right) &= \lim_{n\to\infty} n^2\Big(e^{-4x}-4e^{-3x}K_n^\beta(e^{-t};x)+6e^{-2x}K_n^\beta(e^{-2t};x) \\ &\quad -4e^{-x}K_n^\beta(e^{-3t};x)+K_n^\beta(e^{-4t};x)\Big) \\ &= 3e^{-4x}x^2(1+x)^2.\end{aligned} \tag{3.9}$$

Also, Lemma 2 gives that

$$\lim_{n\to\infty} n^2\mu_{n,4}^\beta(x) = 3x^2(x+1)^2. \tag{3.10}$$

Since, we know that $\lim_{\delta\to 0}\omega^*(f,\delta)=0$, therefore the proof is over by taking limit $n\to\infty$ on both sides of the inequality (3.1), along with the equations (3.9) and (3.10). □

An interesting approach to the convergence of linear positive operators presents Grüss–Voronovskaya-type theorems regarding a so-called multiplicative law for the operators.

Theorem 6 *Assume that the functions f, g and $fg \in C_*(\mathbb{R}_0)$ such that f', g', $(fg)'$, f'', g'' and $(fg)'' \in C_*(\mathbb{R}_0)$. Then, for each $x \in \mathbb{R}_0$, we have*

$$\lim_{n\to\infty} n\left\{K_n^\beta((fg);x) - K_n^\beta(f;x)K_n^\beta(g;x)\right\} = x(x+1)f'(x)\,g'(x).$$

Proof In order to obtain the following identity for operators $K_n^\beta(.;x)$ we utilize the product rule for the first and the second derivatives of fg.

$$\begin{aligned}
&K_n^\beta((fg);x) - K_n^\beta(f;x)K_n^\beta(g;x)\\
&= \left\{K_n^\beta((fg);x) - f(x)g(x) - (fg)'(x)K_n^\beta((t-x);x) - \frac{(fg)''(x)}{2!}K_n^\beta((t-x)^2;x)\right\}\\
&- g(y)\left\{K_n^\beta(f;x) - f(x) - f'(x)K_n^\beta((t-x);x) - \frac{f''(x)}{2!}K_n^\beta((t-x)^2;x)\right\}\\
&- K_n^\beta(f;x)\left\{K_n^\beta(g;x) - g(x) - g'(x)K_n^\beta((t-x);x) - \frac{g''(x)}{2!}K_n^\beta((t-x)^2;x)\right\}\\
&+ \frac{1}{2!}K_n^\beta((t-x)^2;x)\left\{g(x)\,g''(x) + 2f'(x)\,g'(x) - g''(x)K_n^\beta(f;x)\right\}\\
&+ K_n^\beta((t-x);x)\left\{f(x)\,g'(x) - g'(x)K_n^\beta(f;x)\right\}.\\
&=: I_1 - I_2 - I_3 + I_4 + I_5.
\end{aligned}$$

Consequently

$$\lim_{n\to\infty} n(K_n^\beta((fg);x) - K_n^\beta(f;x)K_n^\beta(g;x)) = \lim_{n\to\infty} n(I_1 - I_2 - I_3 + I_4 + I_5). \tag{3.11}$$

By using Theorem 5, we get

$$\lim_{n\to\infty} n(I_1 - I_2 - I_3) = 0. \tag{3.12}$$

Following the estimates of Lemma 2 and Theorem 4, it is easy to see that

$$\begin{aligned}
\lim_{n\to\infty} nI_4 &= \lim_{n\to\infty} \frac{n}{2}K_n^\beta((t-x)^2;x)\left\{g(x) - K_n^\beta(f;x)\right\}g''(x) + f'(x)g'(x)\\
&\quad\times \lim_{n\to\infty} nK_n^\beta((t-x)^2;x)\\
&= x(x+1)f'(x)g'(x),
\end{aligned} \tag{3.13}$$

and

$$\lim_{n\to\infty} nI_5 = \lim_{n\to\infty} nK_n^\beta((t-x);x)\left\{f(x) - K_n^\beta(f;x)\right\}g'(x) = 0. \tag{3.14}$$

Finally, combining (3.12)-(3.14) with (3.11), we get the required result. □

4 Estimation of operator differences

In this section, we will consider the difference between two operators. It means, we estimate the difference $|K_n^\beta - \overline{V}_n|$, where $K_n^\beta(.;x)$ is our new operator and $\overline{V}_n(.;x)$ is the

Kantorovich version of the Baskakov operators. We have chosen $\overline{V}_n(.;x)$ because our new operators $K_n^{\beta}(.;x)$ are connected with Baskakov basis functions.

The main tool, which we shall apply, is Theorem 2.1 from paper [12]. So, first we define the operators $K_n^{\beta}(.;x)$ and $\overline{V}_n(.;x)$ according to the environment of Theorem 2.1.

Let us consider for $f \in C_b(\mathbb{R}_0)$ and $x \in \mathbb{R}_0$ the following form of the operators $K_n^{\beta}(.;x)$:

$$K_n^{\beta}(f;x) = \sum_{j=0}^{\infty} s_j(\beta x) J_{n,j}(f),$$

where, the basis function

$$s_j(\beta x) = e^{-\beta x} \frac{(\beta x)^j}{j!},$$

and the functional

$$J_{n,j}(f) = \left(\frac{n}{x}\right)^{n+j} \frac{1}{\Gamma(n+j)} \int_0^{\infty} \left[e^{-\frac{nt}{x}} t^{n+j-1} \sum_{k=0}^{\infty} e^{-nt} \frac{(nt)^k}{k!} H_{n,k}(f) \right] dt.$$

Similarly, for the operators $\overline{V}_n(.;x)$, we define

$$\overline{V}_n(f;x) = \sum_{j=0}^{\infty} v_{n,j}(x) H_{n,j}(f),$$

where, the basis function and the functional are defined respectively as:

$$v_{n,j}(x) = \binom{n+j-1}{j} \frac{x^j}{(1+x)^{j+n}},$$

$$H_{n,j}(f) = n \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(s) ds.$$

Moreover, we denote by

$$b^{H_{n,j}} = H_{n,j}(e_1), \quad \mu_2^{H_{n,j}} = H_{n,j}(e_1 - b^{H_{n,j}} e_0),$$

$$b^{J_{n,j}} = J_{n,j}(e_1), \quad \mu_2^{J_{n,j}} = J_{n,j}(e_1 - b^{J_{n,j}} e_0).$$

Remark 4 Based on paper [15], we can write for $\overline{V}_n(.;x)$

$$b^{H_{n,j}} = \frac{j}{n} + \frac{1}{2n}, \quad \mu_2^{H_{n,j}} = \frac{1}{12n^2}.$$

Remark 5 For operators $K_n^{\beta}(.;x)$, we can compute similar quantities,

$$b^{J_{n,j}} = \left(\frac{n}{x}\right)^{n+j} \frac{1}{\Gamma(n+j)} \int_0^{\infty} \left[e^{-\frac{nt}{x}} t^{n+j-1} \sum_{k=0}^{\infty} e^{-nt} \frac{(nt)^k}{k!} H_{n,k}(e_1) \right] dt,$$

by Remark 4, we get

$$b^{J_{n,j}} = \left(\frac{n}{x}\right)^{n+j} \frac{1}{\Gamma(n+j)} \int_0^\infty \left[e^{-\frac{nt}{x}} t^{n+j-1} \sum_{k=0}^{\infty} e^{-nt} \frac{(nt)^k}{k!} \left(\frac{k}{n} + \frac{1}{2n}\right)\right] dt.$$

Using the definition of Szász–Mirakyan operators, we obtain

$$b^{J_{n,j}} = \left(\frac{n}{x}\right)^{n+j} \frac{1}{\Gamma(n+j)} \int_0^\infty \left[e^{-\frac{nt}{x}} t^{n+j-1} \left(S_n(e_1;t) + \frac{1}{2n} S_n(e_0;t)\right)\right] dt.$$

By the definition of the Gamma function, we can write

$$b^{J_{n,j}} = \frac{x}{n} \frac{\Gamma(n+j+1)}{\Gamma(n+j)} + \frac{1}{2n} = x\left(1 + \frac{j}{n}\right) + \frac{1}{2n}.$$

Using the previous quantity $b^{J_{n,j}}$ and the definition of $\mu_2^{H_{n,j}}$, we get

$$\mu_2^{H_{n,j}} = \frac{(x+x^2)}{n}\left(1 + \frac{j}{n}\right) + \frac{1}{12n^2}.$$

Now, we are in a position to formulate the difference approximation theorem.

Theorem 7 *If $D(\mathbb{R}_0)$ is the set of all $f \in C(\mathbb{R}_0)$ such that $K_n^\beta(f), \overline{V}_n(f) \in C(\mathbb{R}_0)$ and $f'' \in C_b(\mathbb{R}_0)$, then for $f \in D(\mathbb{R}_0)$, $x \in \mathbb{R}_0$ and $\beta > 0$, the following estimation holds:*

$$\left|(K_n^\beta - \overline{V}_n)(f;x)\right| \le \frac{1}{2}\left[\frac{1}{6n^2} + \frac{x(x+1)^2}{n}\right] \|f''\| + 2\omega\left(f, \frac{\sqrt{4\beta x^2(\beta x^2 + x + 1) + 1}}{2n}\right) + 2\omega\left(f, \frac{\sqrt{4nx(x+1)+1}}{2n}\right).$$

Proof By Theorem 2.1 [12], we have

$$\left|(K_n^\beta - \overline{V}_n)(f;x)\right| \le \alpha(x)\|f''\| + 2\omega(f, \delta_1(x)) + 2\omega(f, \delta_2(x)),$$

where

$$\alpha(x) = \frac{1}{2}\sum_{j=0}^{\infty}\left(s_j(\beta x)\mu_2^{J_{n,j}} + v_{n,j}(x)\mu_2^{H_{n,j}}\right),$$

$$\delta_1^2(x) = \sum_{j=0}^{\infty} s_j(\beta x)\left(b^{J_{n,j}} - x\right)^2, \quad \delta_2^2(x) = \sum_{j=0}^{\infty} v_{n,j}(x)\left(b^{H_{n,j}} - x\right)^2.$$

Using definitions of $s_j(\beta x)$, $v_{n,j}(x)$, Remark 4 and Remark 5, we may write

$$\alpha(x) = \\ = \frac{1}{2}\sum_{j=0}^{\infty}\left(e^{-\beta x}\frac{(\beta x)^j}{j!}\left[\frac{1}{12n^2} + \frac{x^2+x}{n}\left(1 + \frac{j}{n}\right)\right] + \binom{n+j-1}{j}\frac{x^j}{(1+x)^{j+n}}\frac{1}{12n^2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{6n^2} + \frac{x(x+1)^2}{n}\right].$$

For $\delta_1^2(x)$, we get

$$\delta_1^2(x) = \sum_{j=0}^{\infty} e^{-\beta x}\frac{(\beta x)^j}{j!}\left(\frac{xj}{n} + \frac{1}{2n}\right)^2 = \frac{4\beta x^2(\beta x^2 + x + 1) + 1}{4n^2},$$

and for $\delta_2^2(x)$, we achieve

$$\delta_2^2(x) = \sum_{j=0}^{\infty}\binom{n+j-1}{j}\frac{x^j}{(1+x)^{j+n}}\left(\frac{j}{n} + \frac{1}{2n} - x\right)^2 = \frac{4nx(x+1)+1}{4n^2}.$$

The theorem is proved. □

5 Discussion and conclusion

Several techniques are available in the literature to obtain new approximation operators. Let us recall the well-known paper of Ismail and May ([17]), which deals with so-called exponential operators or the paper of Wachnicki and Tyliba ([22]), where the authors introduced semi-exponential Szász–Mirakyan and Weierstrass operators. Later on, new approaches appear, for example see [9] and the references therein. One of them is a composition of operators. Recently, many articles highlighting the benefits of composition techniques have been published. This motivates us to work in this area. The present article provides a new sequence of positive linear operators that results from the composition of two operators. We introduce the moment-generating function and utilize it to compute moments of certain orders. Regarding the approximation theorems, we establish fundamental approximation theorems, error estimation theorem, as well as Voronovskaya-type and Grüss–Voronovskaya-type theorems. In the last section of the paper, the difference estimates of our newly defined composition operators and a Baskakov–Kantorovich-type composition operators are also provided.

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