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Influence of Rail Track Foundation Parameters on the Nonlinear Dynamic Response of a Railway Track

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Abstract

The dynamic response of railway tracks is a key factor influencing the operational safety and reliability of rail transport. Classical analytical methods for modelling track dynamics become insufficient at higher operating speeds, as they typically assume linear behaviour and cannot account for nonlinearities present in the fastening system or in the rail track foundation response. This increases the risk of damage, leading to traffic interruptions, financial losses, and reduced safety. To support predictive maintenance, it is necessary to develop databases based on in-situ measurements, complemented with synthetic data obtained from validated analytical and semi-analytical models. This paper presents such a model, designed to analyse how the parameters of the track foundation – including stiffness and damping – affect the track's dynamic response to loads generated by a moving railway vehicle. The model incorporates experimentally confirmed nonlinear stiffness of the fastening system, represented by a viscoelastic layer that provides continuous support for the rails.

Keywords: rail track dynamics, nonlinear stiffness, analytical modelling, semi-analytical methods

1. Introduction

Modern methods used for modelling railway track dynamics are mainly numerical or semi-numerical. These approaches significantly broaden the ability to analyse complex systems, especially in situations where direct measurement is difficult due to limited accessibility or the lack of sufficiently precise instrumentation. They also allow the study of the influence of external factors and the evaluation of optimisation scenarios for track structure design.

However, numerical approaches are often overused. Their results may be burdened with uncertainties resulting from the internal mechanisms of commercial software, which frequently operates as a "black box". Incomplete knowledge of the modelled system, together with insufficient verification of the assumptions used, may lead to unreliable results. Therefore, validation of numerical simulations should be conducted using experimental measurements or simplified analytical models based on mathematical physics (Timoshenko, 1926; Fryba, 1972; Shenga, Jones and Thompson, 2004; Bogacz and Czyczula, 2008; Czyczula, Koziol and Kudla et al., 2017). Analytical solutions remain an invaluable reference due to their transparency and mathematical accuracy (Mathews, 1958; Koziol and Pilecki, 2020; Koziol and Pilecki, 2021; Koziol, 2023).

When a closed-form analytical solution is not available, an approximation may be formulated so that the result remains sufficiently close to the exact solution, whose existence is ensured by theory. Various non-numerical approximation techniques can be applied depending on the nature of the problem. Their use often requires high-performance computing tools; therefore, it is desirable to simplify the model by considering the essential characteristics of the analysed system. This heuristic approach begins with a clear formulation of the research question. Then it focuses on adapting an existing model or constructing a new one, ensuring that the simplifications still address the research objective without requiring a full mathematical and physical description of the system. Approximate solutions remain theoretically reliable in ranges inaccessible to experimental verification due to their grounding in mathematical reasoning (Koziol, 2010; Koziol and Mares, 2010; Koziol and Hryniewicz, 2012; Czyczula, Koziol and Kudla et al., 2017; Koziol, 2020). Selecting an appropriate computational method is therefore crucial, and approximate methods should not be replaced uncritically by numerical simulations. The building of hybrid models provides new possibilities for the analysis of railway track dynamics, especially in cases of considerations not only theoretical, but also supported by measurements made on real structures, which is reflected in the latest research (Xie, Huang and Zeng et al., 2020; Lasisi and Attoh-Okine, 2021; Qu, Yang and Zhu et al., 2021; Fathi, Mehravar and Rahman, 2023; Wang, Bai and Liu, 2023; Ramos, Correia and Nasrollahi et al., 2024; Xin, Wang and Wang et al., 2024; Zadeh, Edwards and de O. Lima et al., 2024; Sun, Seyedkazemi and Nguyen et al., 2025; Zhai, Stichel and Ling, 2025).

Another important issue is the applicability range of the adopted model. In this study, experimental verification is applied where possible, together with an analysis of the model's physical properties that extend beyond the primary research question. Determining the applicability range requires parametric studies, which may utilise various computational tools. A relevant example is the wavelet based solution for a multilayer continuous rail model described by reduced beam equations, used in the analysis of nonlinear system characteristics (Koziol, 2016; Koziol and Pilecki, 2020; Koziol and Pilecki, 2021; Koziol, Dimitrovova and Pilecki, 2021).

This paper applies such an approach to a two layer double beam model used to analyse the vertical dynamic response of a railway track. The system is solved using a semi analytical approach based on wavelet approximation of the Fourier

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transform. The model includes experimentally confirmed nonlinear stiffness of the fastening system, represented as an additional viscoelastic layer providing continuous support of the rails.

2. Two-layer nonlinear model

The simplest model describing the vertical vibrations of a railway track during the passage of a train is a beam resting on an elastic foundation. This equation may contain additional terms characterizing the mechanical properties of the system, such as the nonlinear stiffness of the foundation.

$$EI_r \frac{\partial^4 w(x,t)}{\partial x^4} + m_r \frac{\partial^2 w(x,t)}{\partial t^2} + c_r \frac{w(x,t)}{\partial t} + k_r w(x,t) + k_{Nr} w^3 = P(x,t) \quad (1)$$

In the equation, the following notation is used: $w(x,t)$ $[\mathrm{m}]$ – vertical vibrations of rails; EI_r $[\mathrm{Nm}^2]$ – bending stiffness of rail steel; m_r $[\mathrm{kg/m}]$ – unit mass of rail; k_r $[\mathrm{N/m}^2]$ – linear stiffness of the rail foundation; c_r $[\mathrm{Ns/m}^2]$ – viscous damping of the rail foundation; $k_{Nr}(x,t)$ $[\mathrm{N/m}^4]$ – nonlinear part of foundation stiffness; $P(x,t)$ $[\mathrm{N/m}]$ – a set of forces generated by axles of train moving uniformly along rails with constant speed $V\,[\mathrm{m/s}]$.

The introduction of nonlinearity necessitates the use of approximations or other specialised computational techniques, typically leading to the linearization of the problem. Since the beams are of infinite length, the condition of radiation at infinity, also known as geometric damping, is naturally satisfied, ensuring that vibrations disappear at a certain distance from the excitation (in the case of a moving train, at a certain distance before and after it). This fact can be one of the criteria for the correctness of the system solved using selected approximation techniques (Koziol, 2010).

A more complex system is the so-called multi-layer model, which, in the case of a railway track, takes the form of a two-layer system:

$$\begin{aligned} &EI_r \frac{\partial^4 u}{\partial x^4} + m_r \frac{\partial^2 u}{\partial t^2} + c_r \left(\frac{\partial u}{\partial t} - \frac{\partial w}{\partial t} \right) + k_r (u - w) = P(x,t) \\ &EI_s \frac{\partial^4 w}{\partial x^4} + m_s \frac{\partial^2 w}{\partial t^2} + c_s \frac{\partial w}{\partial t} + k_s w - c_r \left(\frac{\partial u}{\partial t} - \frac{\partial w}{\partial t} \right) - k_r (u - w) = 0 \end{aligned} \quad (2)$$

$u(x,t)$ $[\mathrm{m}]$ – vertical vibrations of rails; EI_r $[\mathrm{Nm}^2]$ – bending stiffness of rail steel; m_r $[\mathrm{kg/m}]$ – unit mass of rail; k_r $[\mathrm{N/m}^2]$ – linear stiffness of the layer between rails and sleepers (including fastening system); c_r $[\mathrm{Ns/m}^2]$ – viscous damping of the layer between rails and sleepers (including fastening system); $w(x,t)$ $[\mathrm{m}]$ – vertical vibrations of sleepers; m_s $[\mathrm{kg/m}]$ – unit mass of sleepers; k_s $[\mathrm{N/m}^2]$ – linear stiffness of the rail track foundation; c_s $[\mathrm{Ns/m}^2]$ – viscous damping of the rail track foundation; $P(x,t)$ $[\mathrm{N/m}]$ – a set of loads generated by axles of train moving uniformly along rails with constant speed $V\,[\mathrm{m/s}]$.

This model already includes a fastening system, but it is linear, which does not fully reflect the system's real properties. The first equation represents the vertical displacements of the rail, understood as a homogeneous beam, while the second, in the case of a stiffened beam, can represent a concrete slab in a ballastless rail track. To model the response of a conventional pavement, the bending stiffness must be removed from the second equation. Then the bottom layer becomes a rigid body that can represent a longitudinal sleeper, corresponding to a ballasted rail track with classic sleepers. The load must be modelled in an appropriate manner to reflect the rail deflections between the sleepers.

The load modelled in the simplified case by a non-inertial force system contains several components that can be grouped into three categories:

$$Q_k(x,t)=P_S(x,t)+P_D(x,t)+P_R(x,t),\quad P(\tilde{x},t)=\sum_{k=0}^{L-1}Q_k(x,t) \tag{3}$$

There are three components responsible for: the part of the load that is constant in time, generated by the weight of the vehicle (quasi-static solution), the part of the load generated by periodic track irregularities (e.g. deflections between sleepers, wavy wear of the rail running surface, etc.) and the random part, which includes unspecified factors influencing the load variability (these may be, for example, random irregularities of the rail running surface) (Bogacz, Krzyżyński and Popp, 1998; Lombaert, Galvın and Francois et al., 2014; Koziol, 2017; Koziol and Kudla, 2018).

If the random part is omitted, a single load can be written as:

$$Q_k(x,t)=(P_0+\Delta P\cdot\exp(i(\Omega_k t+\varphi_l)))\frac{1}{2a}\cos^2\left(\frac{\pi(x-Vt-s_k)}{2a}\right)H(a^2-(x-Vt-s_k)^2) \tag{4}$$

where: Ω_k – frequency of a single load (axle force); s_k – distances between forces (corresponding to the vehicle axles configuration); φ_l – angular frequency associated with different wheel positions in relation to sleepers at the same moment; V – train speed; $H(^*)$ – Heaviside function (used to describe the wheel-rail contact area); ΔP – additional force generated by regular imperfections (e.g. deflection between sleepers).

The most complex model of this class is the double beam model with two nonlinearities, which, after removing the first term in the second equation, describes ballasted railway track vibrations generated by a moving rail vehicle:

$$\begin{aligned}
&EI_r\frac{\partial^4 u}{\partial x^4}+m_r\frac{\partial^2 u}{\partial t^2}+c_r\left(\frac{\partial u}{\partial t}-\frac{\partial w}{\partial t}\right)+k_r(u-w)+k_{Nr}u^3-k_{Ns}w^3=P(x,t)\\
&EI_s\frac{\partial^4 w}{\partial x^4}+m_s\frac{\partial^2 w}{\partial t^2}+c_s\frac{\partial w}{\partial t}+k_s w-c_r\left(\frac{\partial u}{\partial t}-\frac{\partial w}{\partial t}\right)-k_r(u-w)+k_{Ns}w^3-k_{Nr}u^3=0
\end{aligned} \tag{5}$$

where, additionally: k_{Nr} $[\mathrm{N/m^4}]$ – nonlinear part of stiffness of the layer between rails and sleepers (including fastening system); k_{Ns} $[\mathrm{N/m^4}]$ – nonlinear part of stiffness of the rail track foundation.

Given the computational complexity of the model under consideration, approximations must be used to solve it, primarily due to the presence of nonlinearities. A suitable method is a semi-analytical approach that utilises Adomian's decomposition for nonlinear terms and wavelet approximations based on Coiflet wavelet bases (to determine the inverse Fourier transform and Adomian polynomials (Adomian, 1989; Adomian, 1994; Mallat, 1998; Wazwaz, 1999; Monzon, Beylkin and Hereman, 1999; Wojtaszczyk, 2000; Koziol, 2014). The effectiveness and accuracy of this method have been extensively studied and confirmed in numerous previous publications, making it a valuable tool for solving similar systems, particularly those exhibiting significant response variability under dynamic loads. In such cases, it is challenging to achieve numerical stability of the obtained solutions, and analytical methods do not provide sufficient knowledge to obtain a physically accurate solution. It should be emphasised that the model itself was validated based on comparison with other methods in the linear case, and also by comparison with experimental measurements in the case of taking into account the nonlinear stiffness of the foundation (Koziol, 2010; Koziol, 2016; Czyczula, Koziol and Kudla et al., 2017).

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3. Computational results and preliminary analysis

Due to the high complexity of the approximation, a special computational algorithm was developed, running in the Wolfram Research environment. The example calculations used actual values of the structure and vehicle parameters. The vehicle considered in the analysis is the Falns 441 Va coal wagon (Fig. 1), with four axles configured at distances of 1.8 m, 5.04 m, and 6.84 m. A static load generated by this wagon is equal to $P_0 = 112{,}500\,\mathrm{N}$; $\Delta P = 0.2 \cdot P_0$.

Fig. 1. Falns 441 Va coal wagon (photo by Piotr Tokaj, Railway Institute)

The following ballasted rail track parameters are assumed (Czyczula, Koziol and Kudla et al., 2017):

1. Rail type 60E1: Young's modulus $E = 2.1 * 10^8\,\mathrm{kN/m^2}$, moment of inertia in the vertical plane $I_r = 3055 * 10^{-8}\,\mathrm{m^4}$, unit mass $m_r = 60\,\mathrm{kg/m}$, rail rolling surface without imperfections.
2. Rail foundation (fastening system): $k_r = 8.8 * 10^7\,\mathrm{N/m^2}$, $c_r = 3950\,\mathrm{Ns/m^2}$, $k_{Nr} = 5 * 10^{13}\,\mathrm{N/m^4}$ ($k_{Nr} = 10^{13}\,\mathrm{N/m^4}$);
3. Rail track foundation: $k_s = 8.5 * 10^7\,\mathrm{N/m^2}$ ($k_s = 2.8 * 10^7\,\mathrm{N/m^2}$), $c_s = 81 * 10^3\,\mathrm{Ns/m^2}$, $k_{Ns} = 10^{13}\,\mathrm{N/m^4}$ ($k_{Ns} = 5 * 10^{13}\,\mathrm{N/m^4}$);
4. Unit mass of sleeper: $m_s = 267\,\mathrm{kg/m}$;
5. Train speed: $V = 70\,\mathrm{km/h}$.

Figures 2 and 3 present the maximum vibration response of the rails and sleepers in the vertical direction, given as the complex modulus of the vertical vibration response obtained from the wavelet-based semi-analytical solution. Solid lines denote nonlinear results, while dashed lines represent linear stiffness behaviour.

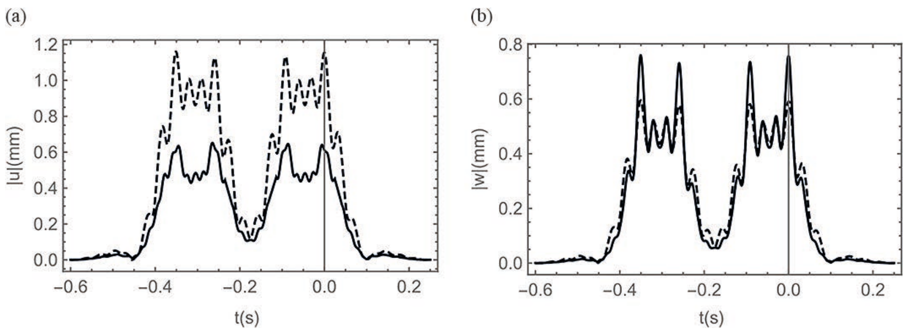

Fig. 2. Maximal response of rail track in good condition – complex modulus of vibrations in vertical direction (solid – nonlinear; dashed – linear; $k_{Nr} > k_{Ns}$): (a) rails; (b) sleepers (own elaboration)

A strong influence of the nonlinearity of the rail fastening system can be observed, especially in the case of a severely degraded track bed. It is worth emphasizing that the considered nonlinearity of the stiffness parameter improves the rail response, measured by the displacement amplitude, while worsening the sleepers dynamics (Figs. 2 and 3).

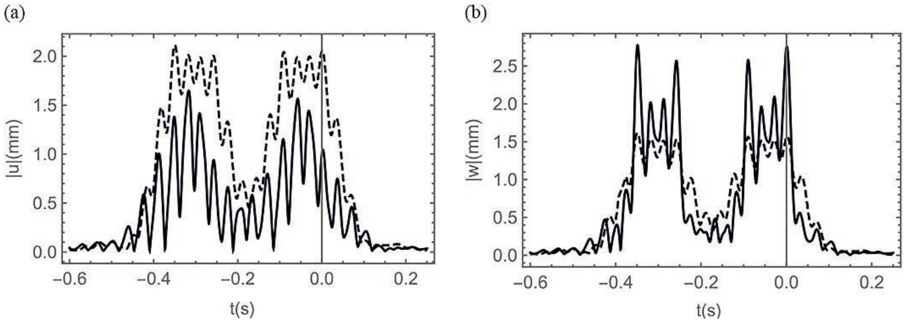

Fig. 3. Maximal response of rail track in bad condition – complex modulus of vibrations in vertical direction (solid – nonlinear; dashed – linear; $k_{Nr} > k_{Ns}$): (a) rails; (b) sleepers (own elaboration)

Degraded track foundation conditions (lower stiffness of the subgrade layer) result in more intense vibration activity in the rails, despite lower displacement amplitudes. This behaviour indicates that the fastening layer and sleepers absorb part of the vibration energy. The nonlinear stiffness of the fastening system alters the energy transfer between layers of the track structure, suggesting that the proper selection of fastening stiffness could be used as an optimisation parameter for predictive maintenance. This observation can be confirmed by examining the energy flow between the track layers, which remains to be investigated in further work.

Furthermore, the results show that in degraded rail track conditions with increased nonlinear stiffness in the foundation (Fig. 4), vibration behaviour changes markedly compared to the well-conditioned track. These effects are

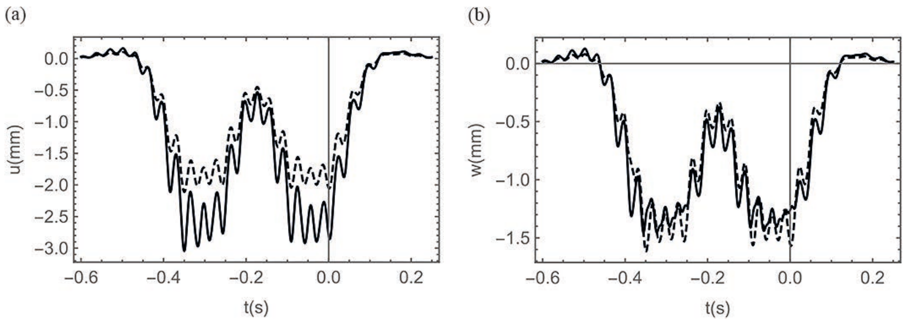

Fig. 4. Vertical vibrations of rail track in bad condition with degraded foundation (solid – nonlinear; dashed – linear; $k_{Nr} < k_{Ns}$): (a) rails; (b) sleepers (own elaboration)

not observed in tracks in good condition. Therefore, it is concluded that the nonlinear mechanical properties of track components play a more significant role in lines with reduced bearing capacity, where the linear stiffness of the track foundation is significantly weakened.

4. Conclusions

The study leads to the following conclusions:

- The proposed multilayer rail track model, combined with the semi--analytical computational method (wavelet-based Fourier approximation), allows efficient analysis of the vertical dynamic response of railway tracks.
- Nonlinear stiffness of the rail fastening system has a significant effect on vibration levels, especially in degraded track conditions.

- Increasing nonlinear fastening stiffness reduces rail displacement amplitudes, but simultaneously intensifies the dynamic response of sleepers.
- The influence of nonlinear foundation stiffness on the rail track response is small or negligible compared to the influence of nonlinear properties of the fastening system.
- The model makes it possible to analyse the effect of track component stiffness on vibration energy distribution, which may support optimisatio of fastening design for vibration control.
- The presented approach is computationally efficient and suitable for large parametric studies, enabling the generation of synthetic datasets useful for predictive maintenance models.

References

Adomian, G. (1989). *Nonlinear Stochastic Systems Theory and Application to Physics*. Dordrecht: Kluwer Academic Publishers.

Adomian, G. (1994). *Solving Frontier Problems of Physics: The Decomposition Method*. Boston: Kluwer.

Monzón, L., Beylkin, G., Hereman, W. (1999). Compactly supported wavelets based on almost interpolating and nearly linear phase filters (coiflets). *Applied and Computational Harmonic Analysis 7*(2), 184–210. https://doi.org/10.1006/acha.1999.0266

Bogacz, R., Czyczula, W. (2008). Response of beam on viscoelastic foundation to moving distributed load. *Journal of Theoretical and Applied Mechanics 46*(4), 763–775.

Bogacz, R., Krzyzynski, T., Popp, K. (1998). Wave propagation in two dynamically coupled periodic systems. In: *Proceedings of the International Symposium on Dynamics of Continua* (pp. 55–64). Bad Honnef: Shaker Verlag.

Czyczuła, W., Chudyba, L., Kapturkiewicz, D., Lisowicz, T. (2023). Nieliniowa aproksymacja oporów systemów przytwierdzeń. *Konferencja Naukowo-Techniczna: Drogi Kolejowe 2023*, Kraków.

Czyczuła, W., Koziol, P., Kudla, D., Lisowski, S. (2017). Analytical evaluation of track response in the vertical direction due to a moving load. *Journal of Vibration and Control 23*(18), 2989–3006. https://doi.org/10.1177/1077546315612120

Fathi, S., Mehravar, M., Rahman, M. (2023). Development of FWD based hybrid back-analysis technique for railway track condition assessment, *Transportation Geotechnics 38*, 100894, ISSN 2214-3912. https://doi.org/10.1016/j.trgeo.2022.100894

Fryba, L. (1972). *Vibration of Solids and Structures under Moving Loads*. Groningen: Noordhoff International Publishing.

Koziol, P. (2010). *Wavelet approach for the vibratory analysis of beam-soil structures: Vibrations of dynamically loaded systems*. Saarbrucken: VDM Verlag Dr. Müller.

Koziol, P. (2014). Wavelet approximation of Adomian's decomposition applied to the nonlinear problem of a double-beam response subject to a series of moving loads. *Journal of Theoretical and Applied Mechanics 52*(3), 687–697.

Koziol, P. (2016). Experimental validation of wavelet based solution for dynamic response of railway track subjected to a moving train. *Mechanical Systems and Signal Processing 79*, 174–181. https://doi.org/10.1016/j.ymssp.2015.03.011

Koziol, P. (2023). Nonlinear "Beam Inside Beam" Model Analysis by Using a Hybrid Semi-analytical Wavelet Based Method. In: *Recent Trends in Wave Mechanics and Vibrations* (pp. 615–621). Springer. https://doi.org/10.1007/978-3-031-23560-3_46

Koziol, P., Dimitrovova, Z., Pilecki, R. (2021). Dynamic properties of a nonlinear double-beam system subjected to a series of moving loads. In: *Proceedings of the 14th World Congress on Computational Mechanics (WCCM XIV) and 8th European Congress on Computational Methods in Applied Science and Engineering (ECCOMAS 2020)*, 2661–2662.

Koziol, P., Kudla, D. (2018). Vertical vibrations of rail track generated by random irregularities of rail head rolling surface. *Journal of Physics: Conference Series 1106, 012007*. https://doi.org/10.1088/1742-6596/1106/1/012007

Koziol, P., Hryniewicz, Z. (2012). Dynamic response of a beam resting on a nonlinear foundation to a moving load: coiflet-based solution. *Shock and Vibration 19*, 995–1007. https://doi.org/10.1155/2012/389476

Koziol, P., Mares, C. (2010). Wavelet approach for vibration analysis of fast moving load on a viscoelastic medium. *Shock and Vibration 17*(4–5), 461–472. https://doi.org/10.1155/2010/579310

Koziol, P., Pilecki, R. (2020). Semi-analytical modelling of multilayer continuous systems nonlinear dynamics. *Archives of Civil Engineering 66*(2), 165–178. https://doi.org/10.24425/ace.2020.131803

Koziol, P., Pilecki, R. (2021). Nonlinear double-beam system dynamics. *Archives of Civil Engineering 67*(2), 337–353. https://doi.org/10.24425/ace.2021.137172

Lasisi, A., Attoh-Okine, N. (2021). Hybrid rail track quality analysis using nonlinear dimension reduction technique with machine learning. *Canadian Journal of Civil Engineering*. https://doi.org/10.1139/cjce-2019-0832

Lombaert, G., Galvin, P., Francois, S., et al. (2014). Quantification of uncertainty in the prediction of railway induced ground vibration due to the use of statistical track unevenness data. *Journal of Sound and Vibration 333*, 4232–4253. https://doi.org/10.1016/j.jsv.2014.03.027

Mallat, S. (1998). A Wavelet Tour of Signal Processing. Academic Press, London. Mathews, P.M. (1958). Vibration of beam on elastic foundation. *Zeitschrift für Angewandte Mathematik und Mechanik 38*, 105–115. https://doi.org/10.1002/zamm.19580380117

Qu, S., Yang, J., Zhu, S., Zhai, W., Kouroussis, G. (2021). A hybrid methodology for predicting train-induced vibration on sensitive equipment in far--field buildings, *Transportation Geotechnics 31*, 100682. https://doi.org/10.1016/j.trgeo.2021.100682

Ramos, A., Correia, A.G., Nasrollahi, K., Nielsen, J.C.O., Calçada, R. (2024). Machine Learning Models for Predicting Permanent Deformation in Railway Tracks, *Transportation Geotechnics 47*, 101289. https://doi.org/10.1016/j.trgeo.2024.101289

Sun, L., Luo, F. (2008). Steady-state dynamic response of a Bernoulli–Euler beam on viscoelastic foundation subjected to a platoon of moving dynamic loads. *Journal of Vibration and Acoustics 130*(051002-17), 1–19. https://doi.org/10.1115/1.2943570

Sun, L., Seyedkazemi, M., Nguyen, C.C., Zhang, J. (2025). Dynamics of Train–Track–Subway System Interaction—A Review, *Machines 13*(11), 1013. https://doi.org/10.3390/machines13111013

Timoshenko, S. (1926). Method of analysis of static and dynamic stresses in rail. In: *Proceedings of the 2nd International Congress on Applied Mechanics* (pp. 407–418). Zurich, Switzerland.

Wang, X., Bai, Y., Liu, X. (2023). Prediction of railroad track geometry change using a hybrid CNN-LSTM spatial-temporal model, *Advanced Engineering Informatics 58*, 102235. https://doi.org/10.1016/j.aei.2023.102235

Wazwaz, A.M. (1999). A reliable modification of Adomian decomposition method. *Applied Mathematics and Computation 102*, 77–86. https://doi.org/10.1016/S0096-3003(98)10191-6

Wojtaszczyk, P. (2000). *Teoria falek. Podstawy matematyczne*. Warszawa: PWN.

Xie, J., Huang, J., Zeng, C., Jiang, S.-H., Podlich, N. (2020). Systematic Literature Review on Data-Driven Models for Predictive Maintenance of Railway

Track: Implications in Geotechnical Engineering, *Geosciences 10*(11), 425. https://doi.org/10.3390/geosciences10110425

Xin, T., Wang, S., Wang, P., Yang, Y., Dai, C. (2024). A Hybrid Method for Vehicle–Track Coupling Dynamics by Analytical and Finite Element Combination Technique, *International Journal of Structural Stability and Dynamics 24*(10), 2450105. https://doi.org/10.1142/S0219455424501050

Zadeh S.S.A., Edwards, J.R., de O. Lima, A., Dersch, M.S., Palma, P. (2024). A data-driven approach to quantify track buckling strength through the development and application of a Track Strength Index (TSI), *Transportation Geotechnics 48*, 101359. https://doi.org/10.1016/j.trgeo.2024.101359

Zhai, W., Stichel, S., Ling, L. (2025). Train–track coupled dynamics problems in heavy-haul rail transportation, Vehicle System Dynamics, International *Journal of Vehicle Mechanics and Mobility 63*(7), 1187–1240. https://doi.org/10.1080/00423114.2025.2494834