Cracow University
of Technology
80th Anniversary

Analytical Model and Design Formulas for Adhesively Bonded Composite Beams

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MEZEROE

OUTLINE

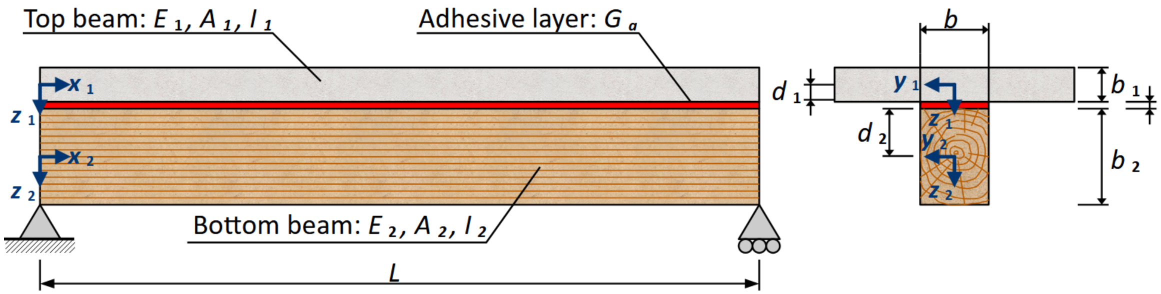
INTRODUCTION

- Composite structures enable **efficient use of advantageous properties of different component materials**.
- One of crucial aspects of **design**, **manufacturing** and **usage** of composite structures is providing appropriate connection between components.
- Adhesive bonding emerges as one of the most efficient method as it enables **connection of practically any pair of materials**.
- Most of European **building codes** such as EC2 (reinforced concrete), EC3 (steel), EC4 (steel-RC composites), CEN-TS 19103 (timber-concrete composites) **do not support design of adhesive joints** with any regulation or guidelines.
  - EC5 (timber structures) deals with gluing technologies only within a limited scope (GLT, LVL, construction of mechanically joint beams, spaced columns and lattice columns)
  - EC9 (aluminum structures) provides general rules of design of adhesively bonded joints transmitting shear force only.
- There is a need of a simple engineering tool for approximate dimensioning of adhesively bonded composite beams.

ANALYTICAL MODEL

ASSUMPTIONS

- linear theory of elasticity:
  - small strains,
  - small displacements,
  - linear constitutive relations



- **bent layers** (panels) **and sheared layers** (adhesive layers) placed in an **alternating way**.
- **bent layers are slender** enough to describe them with the use of Bernoulli-Euler theory.
- **sheared layers** undergo primarily simple shear deformation. This requires that:
  - overall **stiffness of bent layers is considerably greater** than this of sheared layers.
  - **sheared layers are flexible and extensible** – its elongation is of secondary importance.
  - **sheared layers are thin enough** so that deformation state is approximately pure shear.
- **bent panels** may undergo **different axial deformation**, but they share **common deflection**.
- **deformation of sheared layer is determined by deformation of neighboring bent panels**.

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ANALYTICAL MODEL

GOVERNING EQUATIONS are derived from the **equilibrium conditions**

Volkersen shear lag theory

$$\begin{cases} \dfrac{d^2\tilde{u}_1}{d\chi^2} + \beta\left[\left(\tilde{u}_2 - \tilde{u}_1\right) + \alpha\dfrac{d\tilde{w}}{d\chi}\right] = 0 \\ \dfrac{d^2\tilde{u}_2}{d\chi^2} - \gamma\left[\left(\tilde{u}_2 - \tilde{u}_1\right) + \alpha\dfrac{d\tilde{w}}{d\chi}\right] = 0 \\ \dfrac{d^4\tilde{w}}{d\chi^4} = \varepsilon + \delta\left[\left(\dfrac{d\tilde{u}_2}{d\chi} - \dfrac{d\tilde{u}_1}{d\chi}\right) + \alpha\dfrac{d^2\tilde{w}}{d\chi^2}\right] \end{cases}$$

coupling terms

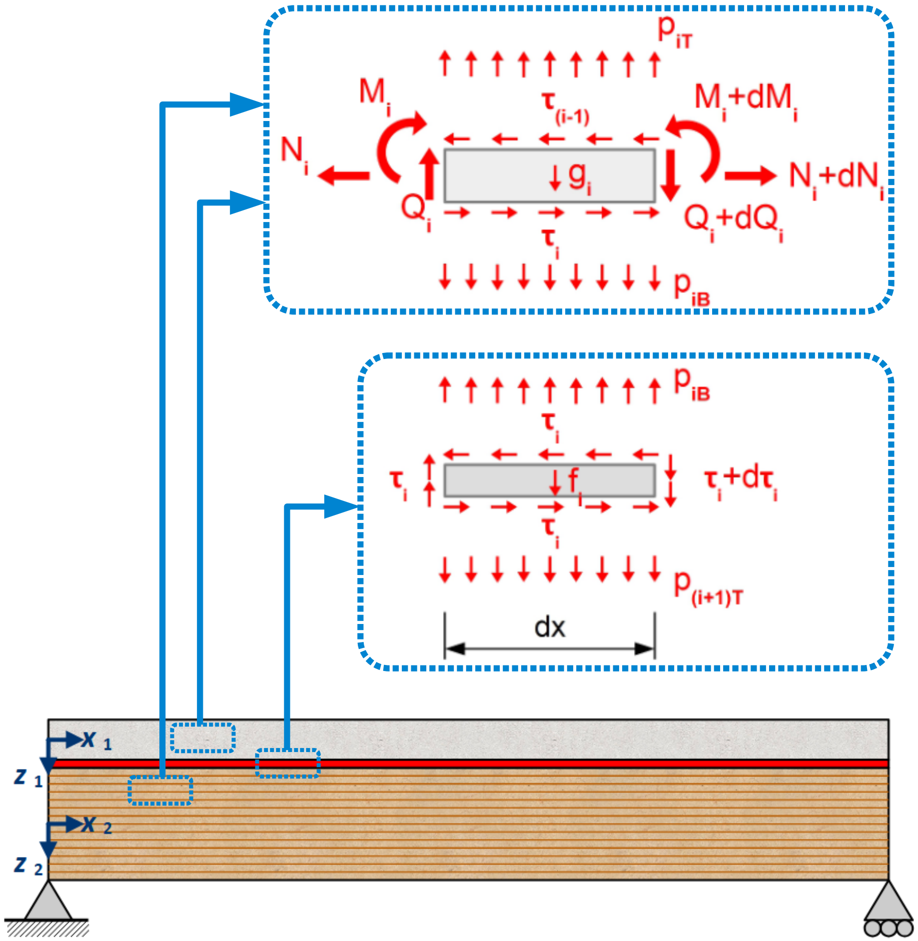
Bernoulli – Euler beam theory

SIMILARITY NUMBERS

$$\alpha = \frac{d_1 + d_2}{L} \qquad \beta = \frac{G_a L^2 b}{E_1 A_1 t} \qquad \gamma = \frac{G_a L^2 b}{E_2 A_2 t}$$

$$\varepsilon = \frac{L^3\left(q + A_1 g_1 + A_2 g_2 + b\,t\,g_a\right)}{E_1 I_1 + E_2 I_2} \qquad \delta = \frac{G_a L^3 b\left(d_1 + d_2 + t\right)}{t\left(E_1 I_1 + E_2 I_2\right)}$$

$$\lambda = \sqrt{\alpha\delta + \beta + \gamma}$$



ANALYTICAL MODEL

CLOSED-FORM SOLUTION is found with the use of the method of generalized eigenvectors

Method of generalized eigenvectors is used when **matrix of coefficients is non-diagonalizable**, namely the **number of linearly-independent eigenvectors (geometric multiplicity m_i) is smaller than algebraic multiplicity k_i of corresponding eigenvalue** λ_i as a root of the characteristic plynomial.

- A generalized eigenvector $\mathbf{v}$ **of rank *r*** is a solution of $\left(\mathbf{A}-\lambda_j\mathbf{1}\right)^{k_j-m_j+1}\mathbf{v}=0$ that satisfies:

$$\left(\mathbf{A}-\lambda_j\mathbf{1}\right)^{r-1}\mathbf{v}\neq 0 \quad \wedge \quad \left(\mathbf{A}-\lambda_j\mathbf{1}\right)^{r}\mathbf{v}=0$$

- It **generates a** chain of generalized eigenvectors according to the following formula:

$$\mathbf{v}_s=\left(\mathbf{A}-\lambda_j\mathbf{1}\right)^{r-s}\boldsymbol{v}, \quad s=1,\cdots,r$$

- **A** collection of chains $\left\{\mathbf{v}_{j,p,1},\mathbf{v}_{j,p,2},\cdots,\mathbf{v}_{j,p,r}\right\}$ **of linearly independent generalized eigevectors** form a basis for a general solution:

$$\mathbf{y}_{j,p,(r-s+1)}=\left(\sum_{s=1}^{r}\frac{x^{r-s}}{(r-s)!}\mathbf{v}_{j,p,s}\right)e^{\lambda_j x}, \quad s=1,2,\cdots,r$$

ANALYTICAL MODEL

CLOSED-FORM SOLUTION

$$\mathbf{y}(\chi) = \mathbf{y}_H(\chi) + \mathbf{y}_N(\chi)$$

general solution of a homogeneous problem (with no external load)

$$\mathbf{y}_H(\chi) = C_1\mathbf{v}_1 e^{\lambda_1\chi} + C_2\mathbf{v}_2 e^{\lambda_2\chi} + [C_3\mathbf{v}_3 + C_4(\chi\mathbf{v}_3 + \mathbf{v}_4)] + \left[C_5\mathbf{v}_5 + C_6(\chi\mathbf{v}_5 + \mathbf{v}_6) + C_7\left(\frac{\chi^2}{2}\mathbf{v}_5 + \chi\mathbf{v}_6 + \mathbf{v}_7\right) + C_8\left(\frac{\chi^3}{6}\mathbf{v}_5 + \frac{\chi^2}{2}\mathbf{v}_6 + \chi\mathbf{v}_7 + \mathbf{v}_8\right)\right]$$

$$\mathbf{v}_1 = [\beta\lambda, \quad -\beta\lambda^2, \quad -\gamma\lambda, \quad \gamma\lambda^2, \quad \delta, \quad -\delta\lambda, \quad \delta\lambda^2, \quad -\delta\lambda^3]^{\mathrm{T}}$$

$$\mathbf{v}_2 = [-\beta\lambda, \quad -\beta\lambda^2, \quad \gamma\lambda, \quad \gamma\lambda^2, \quad \delta, \quad \delta\lambda, \quad \delta\lambda^2, \quad \delta\lambda^3]^{T}$$

$$\mathbf{v}_3 = [1, \quad 0, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0]^{\mathrm{T}}$$

$$\mathbf{v}_4 = [0, \quad 1, \quad 0, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0]^{\mathrm{T}}$$

$$\mathbf{v}_5 = [0, \quad 0, \quad 0, \quad 0, \quad (\beta+\gamma), \quad 0, \quad 0, \quad 0]^{\mathrm{T}}$$

$$\mathbf{v}_6 = [\alpha\beta, \quad 0, \quad -\alpha\gamma, \quad 0, \quad 0, \quad (\beta+\gamma), \quad 0, \quad 0]^{\mathrm{T}}$$

$$\mathbf{v}_7 = [0, \alpha\beta, \quad 0, \quad -\alpha\gamma, \quad 0, \quad 0, \quad (\beta+\gamma), \quad 0]^{\mathrm{T}}$$

$$\mathbf{v}_8 = [\alpha, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0, \quad (\beta+\gamma)]^{\mathrm{T}}$$

particular solution of an non-homogeneous problem (with external load)

$\chi) + \mathbf{y}_N(\chi)$ is found according to the prescribed external load. For UDL:

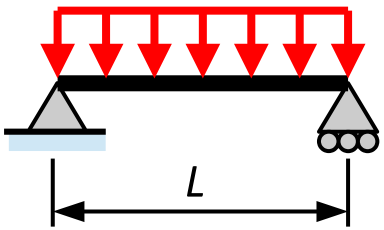
$$\mathbf{y}_N(\chi) = \frac{\varepsilon}{24\lambda}\begin{bmatrix} 4\alpha\beta\chi^3 \\ 12\alpha\beta\chi^2 \\ -4\alpha\chi(6+\gamma\chi^2) \\ -12\alpha\chi(2+\gamma\chi^2) \\ (\beta+\gamma)\chi^4 \\ 4(\beta+\gamma)\chi^3 \\ 12(\beta+\gamma)\chi^2 \\ 24(\beta+\gamma)\chi \end{bmatrix}$$

Constants of integration are determined according to the given boundary conditions

SOLUTION - example

SIMPLY SUPPORTED BEAM UNDER UNIFORMLY DISTRIBUTED LOAD

Boundary conditions:



$$w(0)=0,\quad N_1(0)=0,\quad N_2(0)=0,\quad M(0)=0,$$

$$\phi\left(\frac{L}{2}\right)=0,\quad u_1\left(\frac{L}{2}\right)=0,\quad u_2\left(\frac{L}{2}\right)=0,\quad Q\left(\frac{L}{2}\right)=0$$

Constants of integration:

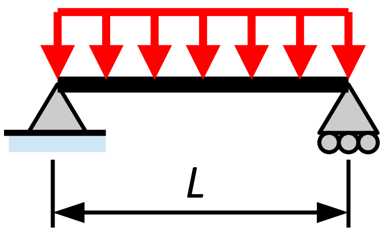
$$C_1=\frac{\alpha\varepsilon e^{\lambda}}{\lambda^6(e^{\lambda}+1)},\quad C_2=\frac{\alpha\varepsilon}{\lambda^6(e^{\lambda}+1)},\quad C_3=\frac{\alpha\gamma\varepsilon}{2\lambda^2(\beta+\gamma)},\quad C_4=\frac{\alpha\beta\varepsilon}{\lambda^2(\beta+\gamma)},$$

$$C_5=-\frac{\alpha\delta\varepsilon}{\lambda^6(\beta+\gamma)},\quad C_6=\frac{\varepsilon\left[\alpha\delta(\beta+\gamma+12)+(\beta+\gamma)^2\right]}{24\lambda^4(\beta+\gamma)},\quad C_7=-\frac{\alpha\delta\varepsilon}{\lambda^4(\beta+\gamma)},$$

$$C_8=-\frac{\varepsilon}{2\lambda^2}$$

DESIGN FORMULAS - example

SIMPLY SUPPORTED BEAM UNDER UNIFORMLY DISTRIBUTED LOAD



Deflection in the middle of the span:

$$w_{\max} = \varepsilon L \frac{768\alpha\delta e^{\frac{\lambda}{2}} + (e^{\lambda}+1)\left[5\lambda^4(\beta+\gamma) + 48\alpha\delta(\lambda^2-8)\right]}{384\lambda^6(e^{\lambda}+1)}$$

Normal stress distribution in the upper adherend in the middle of the span:

$$\sigma_{1,span} = \varepsilon E_1 \left[-\alpha\beta\left(\frac{2e^{\frac{\lambda}{2}}}{\lambda^4(e^{\lambda}+1)} + \frac{\lambda^2-8}{8\lambda^4}\right) + \frac{z_1}{L}\left(\frac{\alpha\delta\left(e^{\frac{\lambda}{2}}-1\right)^2}{\lambda^4(e^{\lambda}+1)} + \frac{\beta+\gamma}{8\lambda^2}\right)\right]$$

Normal stress distribution in the bottom adherend in the middle of the span:

$$\sigma_{2,span} = \varepsilon E_2 \left[\alpha\gamma\left(\frac{2e^{\frac{\lambda}{2}}}{\lambda^4(e^{\lambda}+1)} + \frac{\lambda^2-8}{8\lambda^4}\right) + \frac{z_2}{L}\left(\frac{\alpha\delta\left(e^{\frac{\lambda}{2}}-1\right)^2}{\lambda^4(e^{\lambda}+1)} + \frac{\beta+\gamma}{8\lambda^2}\right)\right]$$

Maximal shear stress in the supported cross-section:

$$\tau_{\max} = \varepsilon\alpha G_a L \frac{\lambda(e^{\lambda}+1) - 2(e^{\lambda}-1)}{2t\lambda^3(e^{\lambda}+1)}$$

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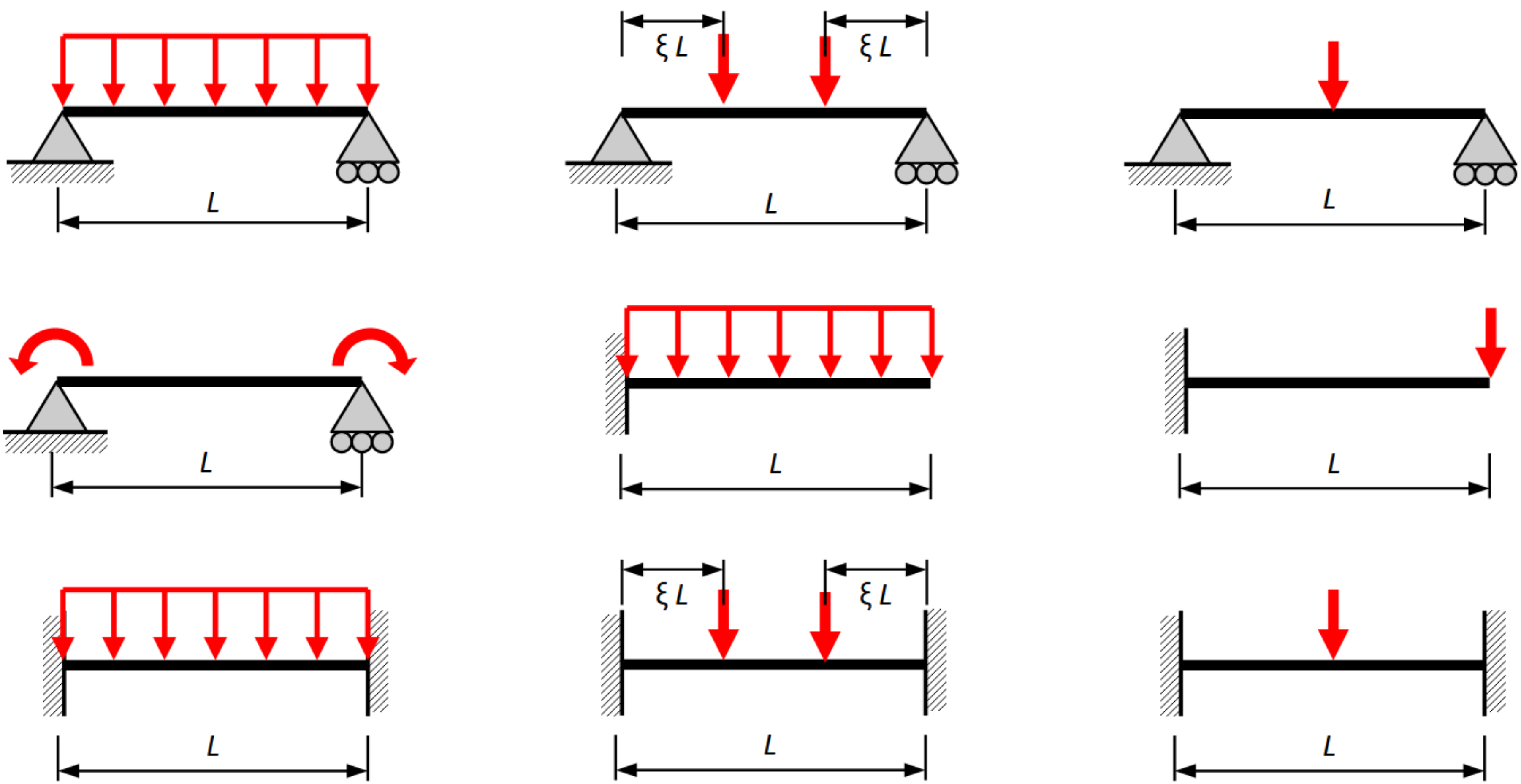
SOLUTION

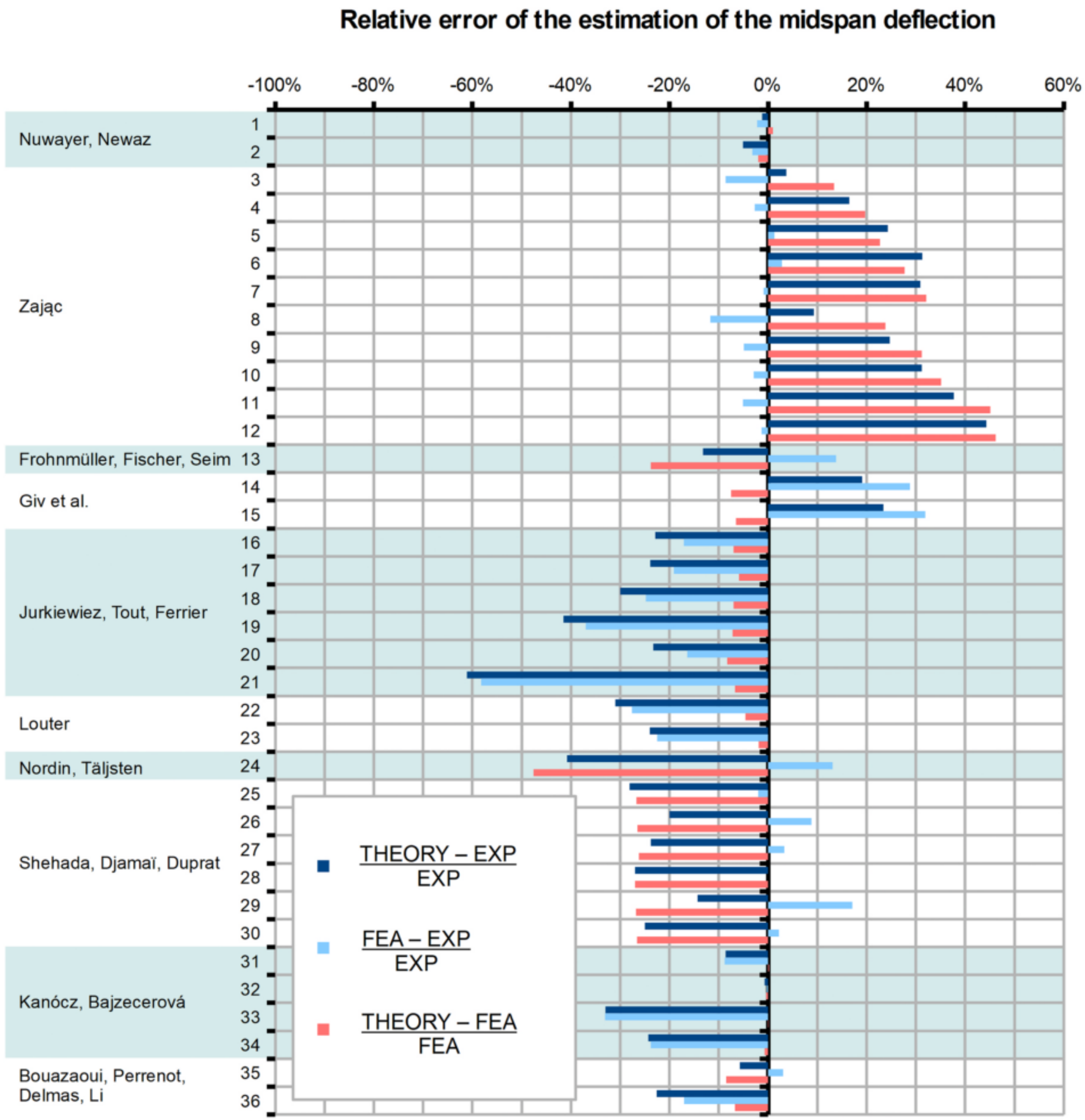
ELABORATED LOAD-SUPPORT LAYOUTS



VALIDATION

- Theoretical predictions were compared with the results of the Finite Element Method simulations as well as with the published experimental results.
- 36 test types conducted on a total of 60 specimens and published in 12 research paper were considered. These included:
  - 3-point bending tests, 4-point bending tests, simple bending tests
  - TCC beams, RC-steel beams, glass-steel beams, metal-metal and CFRP-CFRP composite beams
- 36 FEM models were elaborated. Linear elastic FEA was carried out, assuming full adhesion at the interfaces of the bondline. C3D20 quadratic brick FEs were used. Quarters of bi-symmetric beams were modeled. Simulation was performed with the use of Abaqus/CAE 2022 software.

VALIDATION



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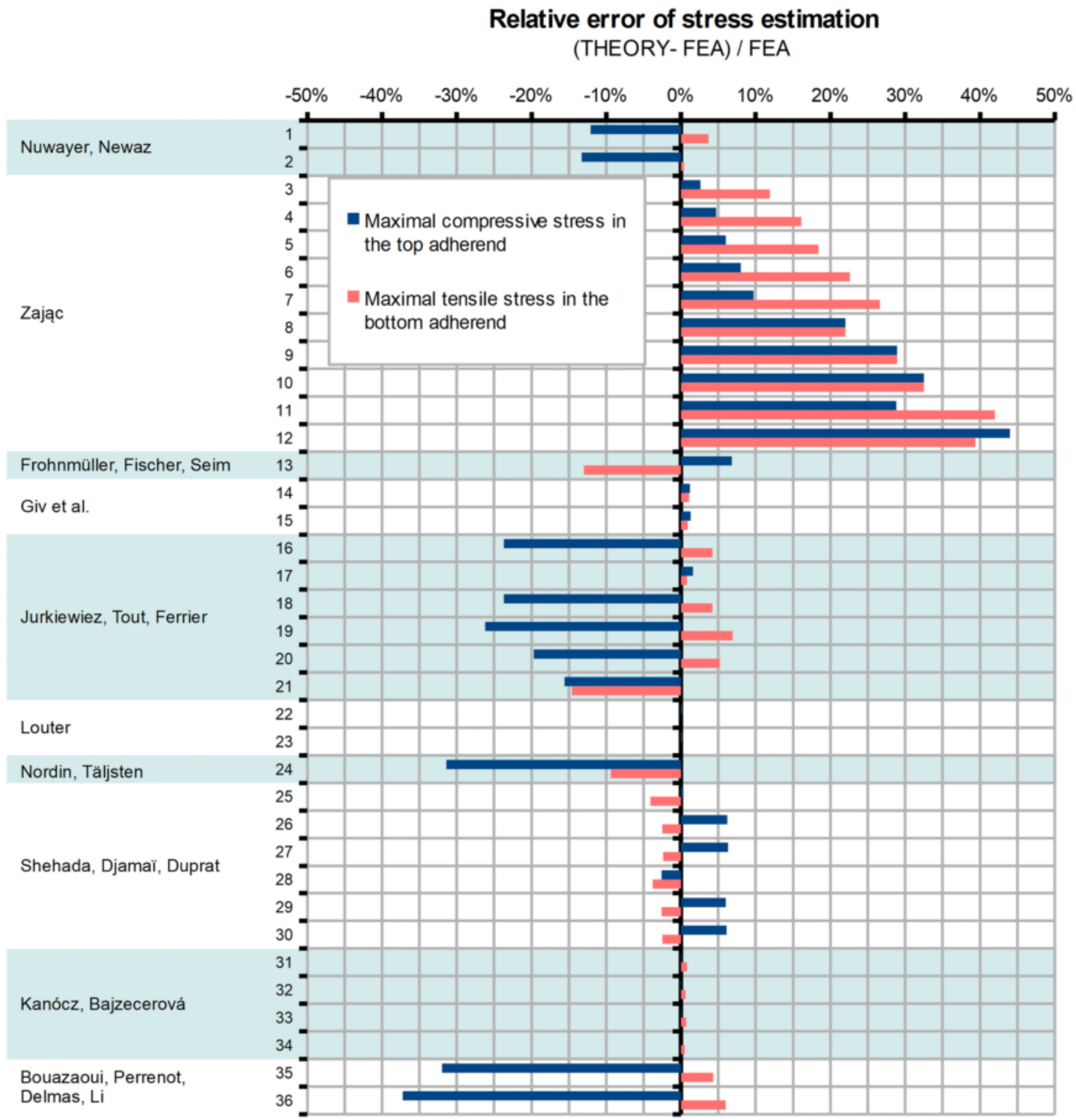
VALIDATION



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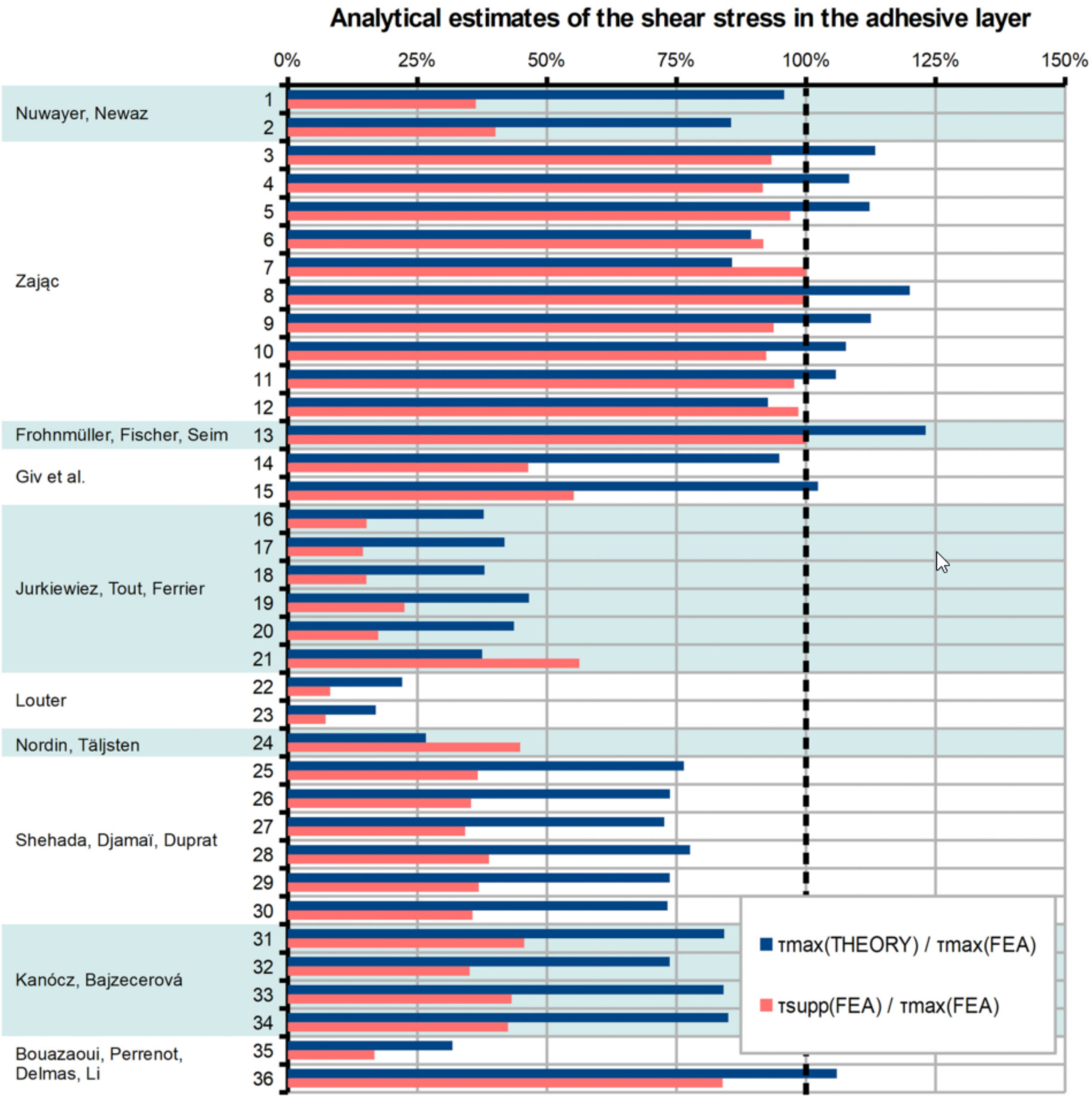

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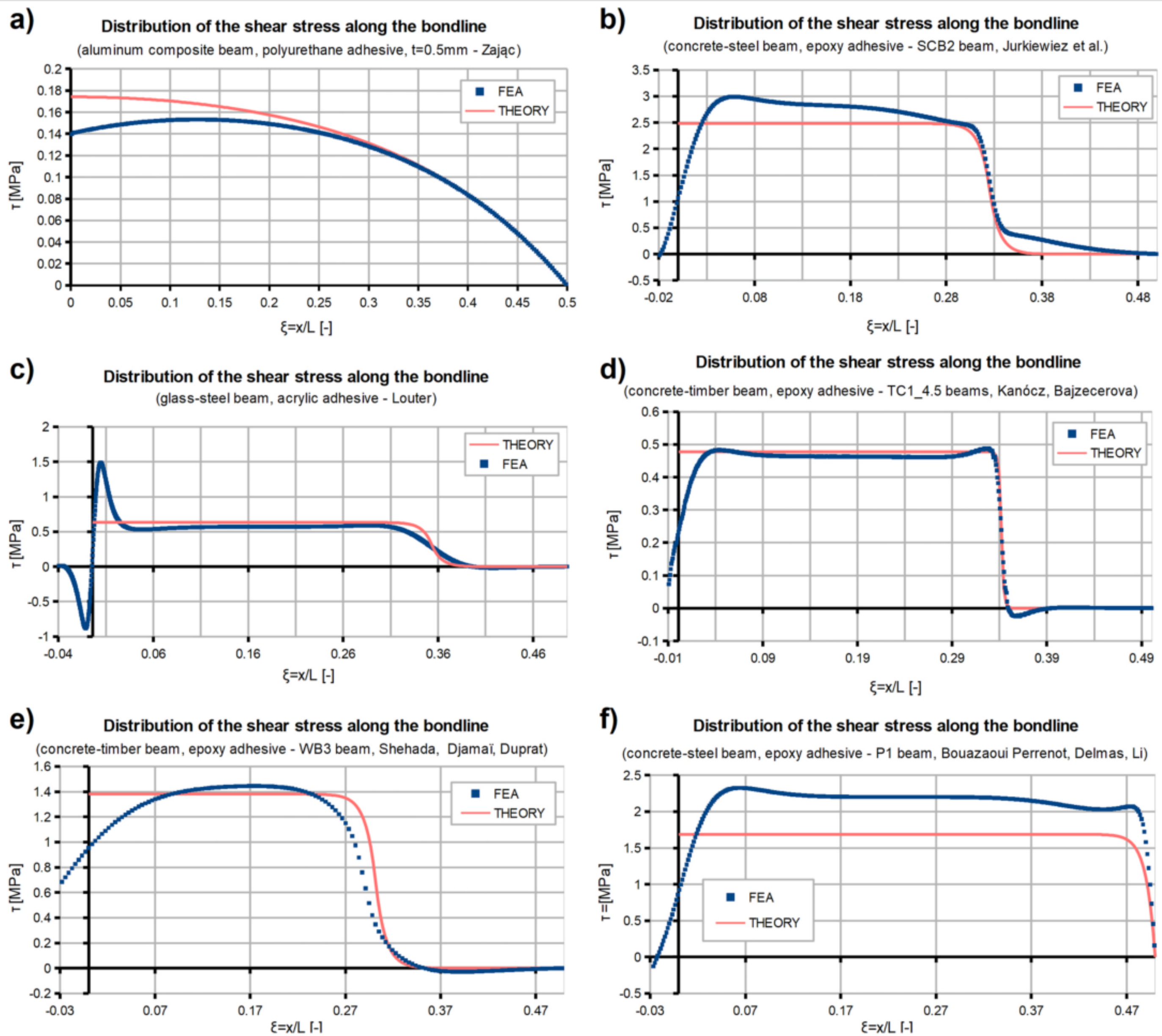

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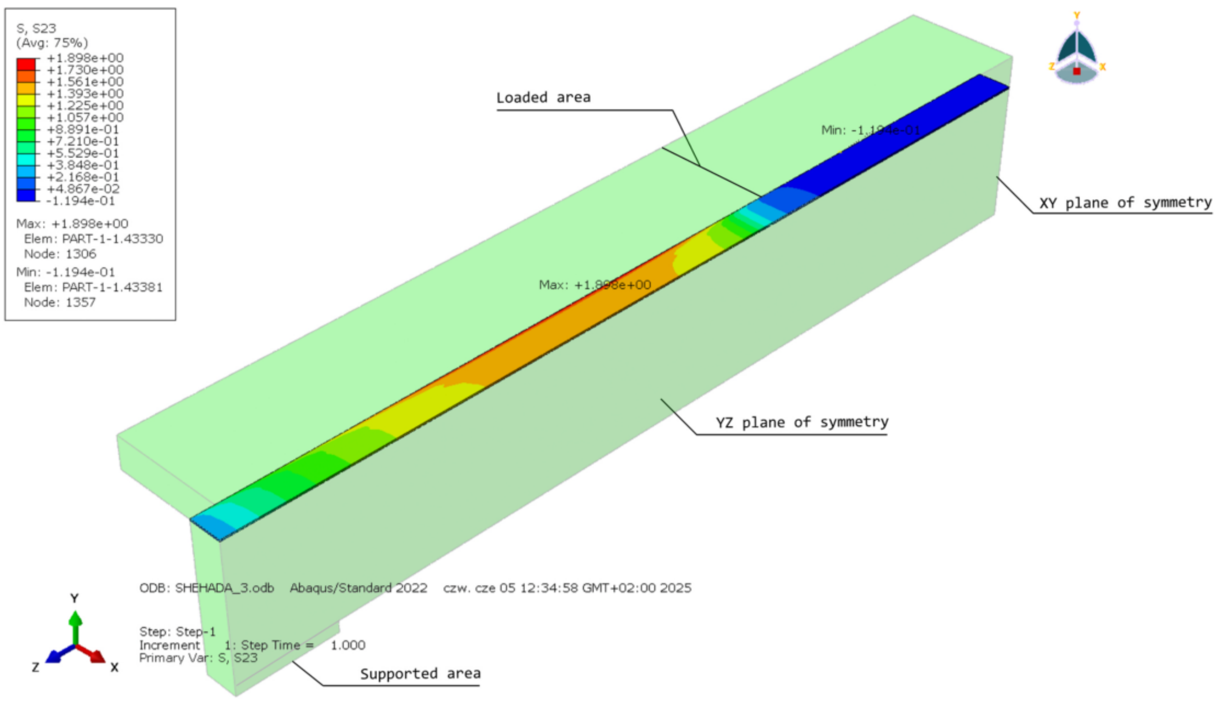
CONCLUSIONS

- Relative error of deflection estimate compared to the experimental result in almost all cases is less than ca. 30%.

- Relative error of normal stress estimate compared to the FEA results in the most of cases in less than ca. 30%

- Design formulas significantly underestimate the extremal shear stress in the bondline. Transverse shear stress distribution and influence of concentrated loads should be investigated.



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CONCLUSIONS

- In an additional research in influence of shear deformation of adherends and longitudinal deformation of the bondline was investigated. It emerged that these factors have negligible influence on the obtained results.

- There are significant computational difficulties in the analysis of the theoretical models of non-symmetric beams.

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