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Explicit periodic solution for the threeparabolic problem in the unbounded strip with boundary conditions of Riquier type

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Abstract

The subject of the paper is the construction of an explicit periodic solution to the boundary problem for a nonhomogeneous threeparabolic equations of sixth order considered in the unbounded space-time layer with boundary conditions of Riquier type. To construct the solution we shall apply the convenient heat iterated Green potentials compatible with boundary condition and the threeparabolic potential compatible with source function.

Keywords: threeparabolic equations, boundary problem, periodic solution, Green function, iterated Green potentials

1. Introduction

The subject of the paper is the construction of an explicit periodic solution u to the equation

$$P^3u(x,t)=f(x,t),\quad P=D_x^2-D_t,\quad P^3=P(P^2),\quad P^2=P(P) \tag{1}$$

in the domain

$$D=\{(x,t): x\in(0,1), t\in(-\infty,\infty)\},$$

satisfying the boundary conditions

$$P^{i-1}u(0,t)=h_i(t),\quad i=1,2,3,\quad t\in(-\infty,\infty) \tag{2}$$

$$P^{i-1}u(1,t)=k_i(t),\quad i=1,2,3,\quad t\in(-\infty,\infty) \tag{3}$$

and the periodicity condition

$$u(x,t+p)=u(x,t),\quad (x,t)\in D, \tag{4}$$

where $p>0$ being the period.

Here functions f, h_i, k_i, $i=1,2,3,$ are given periodic function with period p.

In the following works (Koroński, 1990; 1991; 1995; 1996) explicit solutions of limit problems for three-parabolic equations with different boundary conditions are constructed. In (Koroński, Zdankiewicz, 1996; Koroński, 1999) the periodic solutions of the biparabolic problem with Riquier boundary conditions is considered. In this article the generalization of paper (Koroński, Zdankiewicz, 1996) is given.

2. The Green function

By Krzyżański (1971, p. 476), is know that the Green function for the equation $Pu(x,t)=0,$ for the domain D and boundary conditions of Dirichlet type $G(0,t;y,s)=G(1,t;y,s)=0$ is of form

$$G(x,t;y,s)=Y(t-s)[V_0(x,t;y,s)+H(x,t;y,s)],$$

where

$$V_0(x,t;y,s)=(t-s)^{-\frac{1}{2}}\exp(B(t,s)(x,y)^2),\quad B(t,s)=(-4(t-s)^{-1}),$$

$$H(x,t;y,s)=\sum_{n=1}^{\infty}(-1)^n(V_n^1(x,t;y,s)+V_n^2(x,t;y,s)),$$

$$V_n^1(x,t;y,s)=(t-s)^{-\frac{1}{2}}\exp(B(t,s)(x_n^1,y)^2),$$

$$x_{2n}^1=x+2n,\quad x_{2n+1}^1=-x-2n,\quad n=1,2,3,\ldots,$$

$$V_n^2(x,t;y,s)=(t-s)^{-\frac{1}{2}}\exp(B(t,s)(x_n^2,y)^2),$$

$$x_{2n}^2=x-2n,\quad x_{2n+1}^2=-x+2n+2,\quad n=1,2,3,\ldots,$$

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$Y(t-s)$ denote the Heaviside function, i.e.

$$Y(t-s)=0 \text{ for } t<s \text{ and } Y(t,s)=1 \text{ for } t\geq s.$$

3. The iterated Green potentials compatible with boundary condition

Let us consider following potentials

$$w_1(x,t)=\int_{-\infty}^{t} h_1(s)D_yG(x,t;0,s)ds,$$

$$w_2(x,t)=\int_{-\infty}^{t}\int_{0}^{1}G(x,t;y,s)\int_{-\infty}^{s} h_2(s_1)D_{y_1}G(x,s;0,s_1)ds_1dyds,$$

$$w_3(x,t)=\int_{-\infty}^{t}\int_{0}^{1}G(x,t;y,s)\int_{-\infty}^{s}\int_{0}^{1}G(y,s;y_1,s_1)\int_{-\infty}^{s_1} h_3(s_2)D_{y_2}G(y_1,s_1;0,s_2)ds_2dy_1ds_1dyds,$$

$$w_4(x,t)=\int_{-\infty}^{t} k_1(s)D_yG(x,t;1,s)ds,$$

$$w_5(x,t)=\int_{-\infty}^{t}\int_{0}^{1}G(x,t;y,s)\int_{-\infty}^{s} k_2(s_1)D_{y_1}G(y,s;1,s_1)ds_1dyds,$$

$$w_6(x,t)=\int_{-\infty}^{t}\int_{0}^{1}G(x,t;y,s)\int_{-\infty}^{s}\int_{0}^{1}G(y,s;y_1,s_1)\int_{-\infty}^{s_1} k_3(s_2)D_{y_2}G(y_1,s_1;1,s_2)ds_2dy_1ds_1dyds.$$

4. Properties of the potentials w_i, $i = 1, 2, 3, 4, 5, 6$

Let

$$w(x,t)=\sum_{i=1}^{6} w_i(x,t).$$

Theorem 1. If h_i, k_i, $i=1,2,\dots,6$, are continuous and bounded in $(-\infty,\infty)$, and if h_i, k_i, $i=1,2,\dots,6$, are periodic function, i.e. $h_i(t+p)=h_i(t)$, $k_i(t+p)=k_i(t)$, $i=1,2,\dots,6$, $t\in(-\infty,\infty)$, then

1° $P^3w(x,t)=0$, $(x,t)\in D$,

2° $P^{i-1}w(0,t)=h_i(t)$, $i=1,2,3$, $t\in(-\infty,\infty)$,

3° $P^{i-1}w(1,t)=k_i(t)$, $i=1,2,3$, $t\in(-\infty,\infty)$,

4° $w(x,t+p)=w(x,t)$, $(x,t)\in D$.

Proof. Ad 1°. We have

$$\begin{aligned} Pw(x,t) &= \sum_{i=1}^{6} Pw_i(x,t) \\ &= \int_{-\infty}^{t} h_2(s_1) D_{y_1} G(x,t;0,s) ds_1 + \int_{-\infty}^{t} k_2(s_1) D_{y_1} G(x,t;1,s) ds_1 \\ &+ \int_{-\infty}^{t} \int_{0}^{1} G(x,t;y,s) \int_{-\infty}^{s} h_3(s_1) D_{y_1} G(y,s;0,s_1) ds_1 dy ds \\ &+ \int_{-\infty}^{t} \int_{0}^{1} G(x,t;y,s) \int_{-\infty}^{s} k_3(s_1) D_{y_1} G(y,s;1,s_1) ds_1 dy ds, \end{aligned}$$

$$P^2 w(x,t) = \sum_{i=1}^{6} P^2 w_i(x,t) = \int_{-\infty}^{t} h_3(s_1) D_{y_1} G(x,t;0,s_1) ds_1 + \int_{-\infty}^{t} k_3(s_1) D_{y_1} G(x,t;1,s_1) ds_1,$$

$$\begin{aligned} P^3 w(x,t) &= \sum_{i=1}^{6} P^3 w_i(x,t) \\ &= P \int_{-\infty}^{t} h_3(s_1) D_{y_1} G(x,t;0,s_1) ds_1 + P \int_{-\infty}^{t} k_3(s_1) D_{y_1} G(x,t;1,s_1) ds_1 = 0 \end{aligned}$$

for $(x,t) \in D$.

Ad 2°.

$$\begin{aligned} P^{i-1} w(0,t) &= \sum_{j=1}^{6} P^{i-1} w_j(0,t) \\ &= P^{i-1} \int_{-\infty}^{t} h_1(s) D_y G(0,t;0,s) ds \\ &+ P^{i-1} \int_{-\infty}^{t} \int_{0}^{1} G(0,t;y,s) \int_{-\infty}^{s} h_2(s_1) D_{y_1} G(y,s;0,s_1) ds_1 dy ds \\ &+ P^{i-1} \int_{-\infty}^{t} \int_{0}^{1} G(0,t;y,s) \int_{-\infty}^{s} \int_{0}^{1} G(y,s;y_1,s_1) \int_{-\infty}^{s_1} h_3(s_2) D_{y_2} G(y_1,s_1;0,s_2) ds_2 dy_1 ds_1 dy ds \\ &+ P^{i-1} \int_{-\infty}^{t} k_1(s) D_y G(1,t;0,s) ds + P^{i-1} \int_{-\infty}^{t} \int_{0}^{1} G(1,t;y,s) \int_{-\infty}^{s} k_2(s_1) D_{y_1} G(y,s;0,s_1) ds_1 dy ds \\ &+ P^{i-1} \int_{-\infty}^{t} \int_{0}^{1} G(1,t;y,s) \int_{-\infty}^{s} \int_{0}^{1} G(y,s;y_1,s_1) \int_{-\infty}^{s_1} k_3(s_2) D_{y_2} G(y_1,s_1;0,s_2) ds_2 dy_1 ds_1 dy ds = h_i(t), \end{aligned}$$

$i = 1,2,3, \quad t \in (-\infty, \infty)$.

Ad 3°. Similarly to the 2°, we obtain that

$$P^{i-1} w(1,t) = k_i(t), \quad i = 1,2,3, \quad t \in (-\infty, \infty).$$

Ad 4°. To verify the periodicity of the function $w(x,t)$, we shall prove that the integrals $w_i(x,t)$, $i = 1,2,\ldots,6$ are periodic

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For $w_1(x,t)$ we have

$$w_1(x,t+p)=\int_{-\infty}^{t+p} h_i(s)D_yG(x-0;t+p-s)ds.$$

Applying the change of the integral variable

$$s=p+z,\quad ds=dz,\ \text{ if }\ s\in(-\infty,t+p),\ \text{ then }\ z\in(-\infty,t), \tag{5}$$

we obtain

$$w_1(x,t+p)=\int_{-\infty}^{t} h_i(z+p)D_yG(x-0;t-z)dz$$

and by periodicity h_i we obtain that

$$w_1(x,t+p)=w_1(x,t),\quad (x,t)\in D.$$

For the $w_2(x,t)$ we have

$$w_2(x,t+p)=\int_{-\infty}^{t+p}\int_0^1 G(x-y;t+p-s)\int_{-\infty}^{s} h_2(s_1)D_{y_1}G(y-0;s-s_1)ds_1dyds.$$

Applying the change of the integral variable defined by (5) we obtain

$$w_2(x,t+p)=\int_{-\infty}^{t}\int_0^1 G(x-y;t+p-p-z)\int_{-\infty}^{p+z} h_2(s_1)D_{y_1}G(y-0;p+z-s_1)ds_1dyds.$$

Applying in the integral

$$\int_{-\infty}^{p+z} h_2(s_1)D_{y_1}G(y-0;p+z-s_1)ds_1$$

the change of the integral variable

$$s_1=p+z_1,\quad ds_1=dz_1,\ \text{ if }\ s_1\in(-\infty,p+z),\ \text{ then }\ z_1\in(-\infty,z),$$

we obtain

$$\int_{-\infty}^{p+z} h_2(s_1)D_{y_1}G(y-0;p+z-s_1)ds_1=\int_{-\infty}^{z} h_2(z_1+p)D_{y_1}G(y-0;p+z-p-z_1)dz_1,$$

and by periodicity of h_2, we obtain

$$\begin{aligned}\int_{-\infty}^{p+z} h_2(s_1)D_{y_1}G(y-0;p+z-s_1)ds_1&=\int_{-\infty}^{z} h_2(z_1+p)D_{y_1}G(y-0;z-z_1)dz_1\\&=\int_{-\infty}^{z} h_2(z_1)D_{y_1}G(y-0;z-z_1)dz_1.\end{aligned}$$

Consequently

$$w_2(x,t+p)=w_2(x,t),\quad (x,t)\in D.$$

For the integral $w_3(x,t)$, we have

$$w_3(x,t+p)=\int_{-\infty}^{t+p}\int_0^1 G(x-y;t+p-s)\int_{-\infty}^{s}\int_0^1 G(y-y_1;s-s_1)\int_{-\infty}^{s_1} h_3(s_2)D_{y_2}G(y_1-0;s_1-s_2)ds_2dy_1ds_1dyds.$$

Applying the change of the integral variable

$$s=z+p,\quad ds=dz,\quad \text{if}\quad s\in(-\infty,t+p),\quad \text{then}\quad z\in(-\infty,t),$$

we obtain

$$w_3(x,t+p)=\int_{-\infty}^{t}\int_0^1 G(x-y;t+p-p-z)\int_{-\infty}^{z+p}\int_0^1 G(y-y_1;z+p-s_1)\int_{-\infty}^{s_1} h_3(s_2)D_{y_2}G(y_1-0;s_1-s_2)ds_2dy_1ds_1dyds.$$

Applying in the integral

$$\int_{-\infty}^{z+p}\int_0^1 G(y-y_1;z+p-s_1)\int_{-\infty}^{s_1} h_3(s_2)D_{y_2}G(y_1-0;s_1-s_2)ds_2dy_1ds_1,$$

the change of the integral variable

$$s=z_1+p,\quad ds_1=dz_1,\quad \text{if}\quad s_1\in(-\infty,z+p),\quad \text{then}\quad z_1\in(-\infty,z),$$

we obtain the integral

$$\int_{-\infty}^{z}\int_0^1 G(y-y_1;z+p-z_1-p)\int_{-\infty}^{z_1+p} h_3(s_2)D_{y_2}G(y_1-0;p+z_1-s_2)ds_2dy_1dz.$$

Applying in the integral

$$\int_{-\infty}^{z_1+p} h_3(s_2)D_{y_2}G(y_1-0;p+z_1-s_2)ds_2$$

the change of the integral variable

$$s_2=z_2+p,\quad ds_2=dz_2,\quad \text{if}\quad s_2\in(-\infty,z_2+p),\quad \text{then}\quad z_2\in(-\infty,z_1).$$

By periodicity h_3 by last integral, we obtain

$$\int_{-\infty}^{z_1} h_3(z_2)D_{y_2}G(y_1-0;z_1-z_2)dz_2.$$

Consequently we obtain

$$w_3(x,t+p)=w_3(x,t),\quad (x,t)\in D.$$

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5. Potential compatible with source function ant its properties

Let us consider the potential

$$w_7(x,t)=$$
$$=\int_{-\infty}^{t}\int_{0}^{1}G(z,t;y,s)\int_{-\infty}^{s}\int_{0}^{1}G(z,t;y_1,s_1)\int_{-\infty}^{s_1}\int_{0}^{1}f(y_2,s_2)G(y_1,s_1,y_2,s_2)dy_2ds_2dy_1ds_1dyds.$$

Theorem 2. If the function $f\in C^1(D)\cap B(D)$, where $B(D)$ denote the class of all function bounded in the domain D, and if the function f is periodic with respect to t, i.e. $f(x,t+p)=f(x,t)$, then:

1° $P^3w_7(x,t)=f(x,t),\ (x,t)\in D,$

2° $P^{i-1}w_7(0,t)=0,\ i=1,2,3,\ t\in(-\infty,\infty),$

3° $P^{i-1}w_7(1,t)=0,\ i=1,2,3,\ t\in(-\infty,\infty),$

4° $w_7(x,t+p)=w_7(x,t),\ (x,t)\in D.$

Proof. Ad 1°. By Poisson theorem, (Krzyżański, 1971, p. 522), we obtain

$$Pw_7(x,t)=\int_{-\infty}^{t}\int_{0}^{1}G(z,t;y_1,s_1)\int_{-\infty}^{s_1}\int_{0}^{1}f(y_2,s_2)G(y_1,s_1,y_2,s_2)dy_2ds_2dy_1ds_1.$$

Applying to last integral Poisson theorem we obtain

$$P^2w_7(x,t)=\int_{-\infty}^{t}\int_{0}^{1}f(y_2,s_2)G(x,t;y_2,s_2)dy_2ds_2.$$

Applying once more the Poisson theorem, we obtain

$$P^2w_7(x,t)=f(x,t),\quad (x,t)\in D.$$

Ad 2°. We have

$$w_7(0,t)=$$
$$=\int_{-\infty}^{t}\int_{0}^{1}G(0,t;y,s)\int_{-\infty}^{s}\int_{0}^{1}G(y,s;y_1,s_1)\int_{-\infty}^{s_1}\int_{0}^{1}f(y_2,s_2)\,G(y_1,s_1,y_2,s_2)dy_2ds_2dy_1ds_1dyds=0,$$

$$Pw_7(0,t)=\int_{-\infty}^{t}\int_{0}^{1}G(0,t;y_1,s_1)\int_{-\infty}^{s_1}\int_{0}^{1}f(y_2,s_2)\,G(y_1,s_1,y_2,s_2)dy_2ds_2dy_1ds_1=0,$$

$$P^2w_7(0,t)=\int_{-\infty}^{t}\int_{0}^{1}f(y_2,s_2)\,G(0,t,y_2,s_2)dy_2ds_2=0.$$

Ad 3°. The proof 3° is similar to those 2°.

Ad 4°. We have

$$w_7(x,t+p)=\int_{-\infty}^{t+p}\int_0^1 G(x-y;t+p-s)\int_{-\infty}^{s}\int_0^1 G(y,s;y_1,s_1)\int_{-\infty}^{s_1}\int_0^1 f(y_2,s_2)\,G(y_1,s_1,y_2,s_2)dy_2ds_2dy_1ds_1dyds.$$

Applying the change of the integral variable

$$s=p+z,\quad ds=dz,\quad \text{if}\quad s\in(-\infty,t+p),\quad \text{then}\quad z\in(-\infty,t),$$

we obtain

$$w_7(x,t+p)=\int_{-\infty}^{t}\int_0^1 G(x-y;t+p-p-z)\int_{-\infty}^{z+p}\int_0^1 G(y-y_1;p+z-s_1)\int_{-\infty}^{s_1}\int_0^1 f(y_2,s_2)\,G(y_1,s_1,y_2,s_2)dy_2ds_2dy_1ds_1dyds.$$

Let us consider the integral

$$\int_{-\infty}^{z+p}\int_0^1 G(y-y_1;p+z-s_1)\int_{-\infty}^{s_1}\int_0^1 f(y_2,s_2)\,G(y_1-y_2;s_1-s_2)dy_2ds_2dy_1ds_1.$$

Applying the change of the integral variable

$$s_1=z_1+p,\quad ds_1=dz_1,\quad \text{if}\quad s_1\in(-\infty,p+z),\quad \text{then}\quad z_1\in(-\infty,z),$$

we obtain

$$\int_{-\infty}^{z}\int_0^1 G(y-y_1;p+z-z_1-p)\int_{-\infty}^{z_1+p}\int_0^1 f(y_2,s_2)\,G(y_1-y_2;z_1+p-s_2)dy_2ds_2dy_1ds_1.$$

Let us consider the integral

$$\int_{-\infty}^{z_1+p}\int_0^1 f(y_2,s_2)\,G(y_1-y_2;z_1+p-s_2)dy_2ds_2.$$

Applying the change of the integral variable

$$s_2=z_2+p,\quad ds_2=dz_2,\quad \text{if}\quad s_2\in(-\infty,z_1+p),\quad \text{then}\quad z_2\in(-\infty,z_1),$$

we obtain

$$\int_{-\infty}^{z_1}\int_0^1 f(y_2,z_2+p)\,G(y_1-y_2;z_1+p-z_2+p)dy_2dz_2.$$

By periodicity of the function f, and by the last integral, we obtain

$$\int_{-\infty}^{z_1}\int_0^1 f(y_2,z_2)\,G(y_1-y_2;z_1-z_2)dy_2dz_2.$$

Consequently we obtain periodicity

$$w_7(x,t+p)=w_7(x,t),\quad (x,t)\in D.$$

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6. Conclusion – an explicit periodic solution of the problem (1)–(4)

By Theorem 1 and Theorem 2, we obtain the following result.
Theorem 3. If the assumptions of the Theorem 1 and Theorem 2 are satisfied, then the function

$$w(x,t)=\sum_{i=1}^{7} w_i(x,t)$$

is a periodic solution of (1)–(4) problem.

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